*International Conference on Mathematics and Computational Methods Applied to Nuclear Science and Engineering (M&C 2011)*
Rio de Janeiro, RJ, Brazil, May 8-12, 2011, on CD-ROM, Latin American Section (LAS) / American Nuclear Society (ANS)
ISBN 978-85-63688-00-2

# Opportunities and Challenges in Applying the Compressive Sensing Framework to Nuclear Science and Engineering

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## ABSTRACT

Compressive sensing is a 5-year old theory that has already resulted in an extremely large number of publications in the literature and that has the potential to impact every field of engineering and applied science that has to do with data acquisition and processing. This paper introduces the mathematics, presents a simple demonstration of radiation dose reduction in x-ray CT imaging, and discusses potential application in nuclear science and engineering.

*Key Words*: compressive sensing, nuclear engineering, radiation detection, Monte Carlo simulation, reactor physics, CT dose.

## 1. INTRODUCTION

Compressed sensing or compressive sensing (CS) is an emerging and widely popular mathematical framework that promises nearly perfect signal recovery from a limited number of measurements in situations where the signal is “sparse” or “compressible” [1,2,3,4]. Although some of the underlying concepts of CS have been around for decades, its recent fame is directly attributable to a series of ingenious papers published in the mid 2000s which proved performance bounds on such a recovery [1,2,3]. Armed with what many consider today as one of the most major mathematical discoveries in decades and the broad appeal of “obtaining more with less,” CS has attracted a surprisingly large number of researchers from nearly all fields having to do with signal acquisition and processing. In principle, CS has the potential to benefit engineering applications where one seeks to solve an underdetermined system of linear equations and where the burden associated with a measurement is costly (e.g., expensive instrument, excessive time, or risky side effect). The understanding of the big picture of CS is still rapidly evolving, and information pieces available in the literature and on-line can be overwhelming to starters. One challenge lies with the fact that few individuals possess the ability to bridge the gap between the underlying mathematics and application-specific knowledge, without which one would fail to put these pieces together. As a result, despite the popularity and successful demonstration in computer vision and image processing, the full potential of the CS has yet to be realized. To date, there is limited effort in adopting CS for nuclear science and engineering problems.

The purpose of this paper is to identify both the opportunities and challenges in applying CS to issues important in radiation detection and reactor physics. The mathematical framework of CS is concisely described, which is followed by a computational experiment to reduce radiation dose in x-ray computed tomography (CT). Finally, potential applications are briefly discussed.

## 2. THEORY OF COMPRESSIVE SENSING

Compressive sensing seeks to recover a signal from fewer but highly informative measurements, and uses prior knowledge that the true signal is “sparse” to make up for the small number of measurements. By a “sparse” or compressible signal it is meant that in some basis it can be represented well by keeping only a few large coefficients and discarding many small (i.e. nearly zero) coefficients.

In mathematical terms we aim to find “sparse solutions” to the system of linear equations,

(1) $$\mathbf{Uh} = \mathbf{y},$$

where $\mathbf{U} \in \Re^{m\times n}$ is the measurement or detector matrix, $\mathbf{y} \in \Re^{m}$ are the measurement values, and $\mathbf{h} \in \Re^{n}$ is the discretized signal of interest to be recovered. We consider here a full-ranked, underdetermined system so that the number of measurements *m* is much smaller than the number of unknowns, *n*. Equation (1) then has either no solution or infinitely many solutions. The classical line of approach to solving (1) is to solve the least-squares optimization problem

(2) $$\hat{\mathbf{h}} = \text{argmin} \, \| \mathbf{h} \|_2 \text{ subject to } \mathbf{Uh} = \mathbf{y}.$$

The solution to (2) can be written explicitly as a multiplication of the measurements by the Moore-Penrose right pseudo-inverse of U,

(3) $$\hat{\mathbf{h}} = U^T (UU^T)^{-1} y.$$

However, minimizing the L2 norm (standard Euclidean norm) as suggested in (2) rarely results in “sparse” solutions. Instead, the CS theory suggests that sparse solutions to (1) is available by solving the optimization problem

(4) $$\hat{\mathbf{h}} = \text{argmin} \, \| \mathbf{h} \|_0 \text{ subject to } \mathbf{Uh} = \mathbf{y},$$

where the L2 norm of **h** has now been replaced by the L0 norm (# non-zero components). However, unlike equation (2) there is no explicit solution to (4) and no efficient algorithms for solving (4) have been found to date. The amount of work in solving (4) increases exponentially with the number of unknowns, *n,* and the problem is said to be NP-hard since it can only be solved in non-polynomial time. Instead, CS seeks to solve (4) indirectly by considering

(5) $$\hat{\mathbf{h}} = \text{argmin} \, \| \mathbf{h} \|_1 \text{ subject to } \mathbf{Uh} = \mathbf{y},$$

where the L0 norm has now been replaced by the L1 norm (square root of the sum of the absolute values of each component). Like (4), the optimization problem (5) has no explicit solutions. However, it turns out that unlike (4), (5) is equivalent to a linear program and, as such, efficient algorithms for solving it in polynomial time are available.

CS seeks to solve (4) indirectly by finding sufficient conditions so that the solution to (4) and (5) are the same. These conditions are that the sensing matrix **U** must satisfy the so-called restricted isometry property. That is, for any $0<k<m$ and any *m x s* submatrix of $\mathbf{U_s}$ of **U** with $s \leq k$ we have that

(6)
$$(1-\delta_k) \leq \frac{\| \mathbf{U_s x} \|_2^2}{\| \mathbf{x} \|_2^2} \leq (1+\delta_k)$$

holds for all k-sparse signals **x** ($\|\mathbf{x}\|_0 \leq k$) for some fixed constant $\delta_k \geq 0$ [4]. Equation (6) basically requires that **U** acts in such a way as to preserve the L2 norm of all k-sparse signals. It can be shown that if **U** satisfies the restricted isometry property and that the signal **h** is "sparse enough" then the solution to (4) and (5) are unique and equivalent. "Sparse enough" is defined as **U** satisfying the 2k restricted isometry property with

(7)
$$\delta_{2k} \geq \sqrt{2} - 1$$

for a k-sparse signal **h**.

## 3. COMPRESSIVE SENSING CT DOSE REDUCTION

In this section, an application of the CS theory to the reduction of radiation dose from x-ray CT is illustrated. Nearly all commercially available x-ray CT scanners use an image reconstruction algorithm known as the Filtered Back-Projection (FBP). FBP provides good image quality and fast reconstruction speed. However, restricted by the Shannon sampling theorem, FBP requires hundreds of projection angles to recover the image of an object without introducing artifacts into the reconstructed image data. Taking measurements at hundreds of projection angles means that the patient must be subjected to large amount of radiation dose. While the risk associated with x-ray CT to an individual is generally minimal and medically justified, a recent report by the NCRP has indicated that the number of CT procedures has increased by 20 folds in the past 20 years — a worrisome trend in medical imaging that, if not stopped, would lead to increased cancer incidence among the U.S. population. Therefore, it is attractive to develop methods that would minimize the radiation exposure required to form a CT image.

Here, we perform a computational experiment to compare two reconstruction algorithms based on the FBP and CS method, leading to the conclusion that a CT image can be created with ten times fewer projection angles than FBP while maintaining acceptable image quality. The experiment involves a CT image model called the Shepp-Logan phantom representing the human cerebral which is illustrated in Figure 1.

**Figure 1. Shepp-Logan phantom (256 × 256 pixels)**

Using algorithms implemented in the Matlab software program, we consider a simplified model of CT reconstruction process. The FBP simulations involved several steps of operation including the forward-projection, Fourier transform, filtering, inverse Fourier transform, and back-projection. In this study, the CS-based image reconstruction algorithm was performed using the *Toolbox Sparsity* [5], in which inverse problems are solved with Total Variance (TV) regularization. The TV regularization is achieved by seeking the optimization:

$$\hat{\mathbf{h}} = \arg\min_{\mathbf{h}} \frac{1}{2} \| \mathbf{y} - \mathbf{U}\mathbf{h} \|_2 + \tau \| \mathbf{h} \|_{TV} \quad \text{where} \quad \| \mathbf{h} \|_{TV} = \sum_n \sqrt{| \mathbf{D_1 h}[n] |^2 + | \mathbf{D_2 h}[n] |^2} \qquad (8)$$

Note that $\tau$ is regularization parameter. D1 and D2 are the finite difference operators along horizontal and vertical directions of the image slice. In this light, (8) can be understood as an optimization problem which balances between satisfying the measured data in the L2 (least squares) sense and having a “sparse image gradient” in the L1 norm sense. To computationally solve this optimization problem, Chambolle's algorithm was used [6]. Given the noise level, $\tau$ is automatically updated during the iteration. In the image reconstruction, white Gaussian noise of 20dB signal-noise ratio was added into the partial Fourier measurements in the forward projections.

In addition, the mean square error (MSE), which is an indicator of the error between original and reconstructed images, was calculated using eq. 9.

$$MSE = \frac{1}{WH} \sum_i \sum_j | \tilde{f}(i,j) - h(i,j) |^2 \qquad (9)$$

Table I summarizes MSE for images reconstructed using different number of projections. The results show clearly that, for 288 projections, the FBP algorithm yields an MSE of $8.7 \times 10^{-3}$ while the CS-based TV-regularization algorithm yields an MSE of $1.1 \times 10^{-4}$ which is about 80 times better. As the number of projections decreases, the MSE for both algorithms become worse. However, with 18 projections, the CS-based TV regularization algorithm yields an MSE of $1.1 \times 10^{-3}$ which is still 8 times better than the image reconstructed using 288 projections for the FBP algorithm. This suggests that, using the CS-based TV-regularization algorithm, one can achieve the same image quality with many fewer projections, thus reducing the amount of radiation exposure to the patient. Figure 1 compares the reconstructed images for various number of projection angles. The results show that the FBP algorithm is very sensitive to the number of projections and image artifacts become more obvious and severe as the number of projections decreases. By contrast, the CS TV-regularization algorithm continues to provide good images. If CS based algorithm (such as TV-regularization demonstrated here) can be utilized in commercial CT scanners, the radiation dose to the patient can be reduced by a factor of 10 from the current dose level. It should be noted that the CS reconstruction consumes more computational power than does the traditional FBP. For example, on a standard desktop computer it takes 38 seconds to reconstruct the image using CS TV-regularization involving 288 projections, in comparison with about 0.03 seconds for the FBP algorithm involving the same number of projections. However, acceleration schemes using either hardware such as the graphics processing units (GPUs) or software to solve CS problems would effectively circumvent the computational burden [16].

**Table I. Mean square error (MSE) for varying number of projections.**

| | **Number of Projections** | | | | |
|---|---|---|---|---|---|
| | **288** | **144** | **72** | **36** | **18** |
| **FBP** | $8.7 \times 10^{-3}$ | $8.9 \times 10^{-3}$ | $1.0 \times 10^{-2}$ | $1.8 \times 10^{-2}$ | $4.3 \times 10^{-2}$ |
| **CS TV Regularization** | $1.1 \times 10^{-4}$ | $2.0 \times 10^{-4}$ | $3.0 \times 10^{-4}$ | $5.1 \times 10^{-4}$ | $1.1 \times 10^{-3}$ |

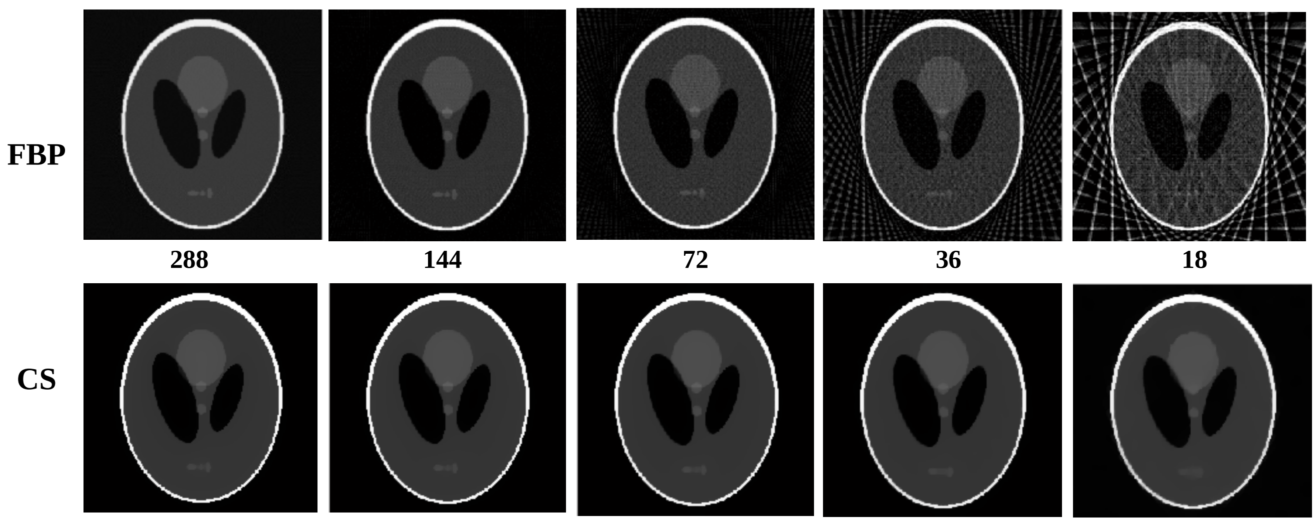


**Figure 2. Comparison of reconstructed images of the Shepp-Logan phantom for different projection angles using two algorithms: Filtered Back Projection (top row) and CS TV-regularization (bottom row).**

## 4. POTENTIAL APPLICATIONS TO NUCLEAR SCIENCE AND ENGINEERING

The CS framework suggests that one no longer has to satisfy the well established Nyquist–Shannon criterion in signal processing [7]. The body of literature, mostly published in 2004-2006, by people such as Donoho, Candès, Romberg and Tao [e.g., 1-3], have set a mathematical foundation that basically predicts the minimum amount of data needed to reconstruct a signal even though the information would be deemed insufficient by the Nyquist–Shannon criterion. One example that probably everyone can relate to is the fact that a modern digital camera gathers huge amounts of information and then compress the images. According to the CS framework, such a compression process is a gigantic waste because the useful information is actually sparse. Today there are plenty of researchers who are attracted to the promise that one can “get more information with less measurement.” Others still wonder whether or not CS is one of those hypes that will fade away. To date, nuclear engineers and scientists have been slow in engaging ourselves in this particular scientific endeavor of our time. Among several challenges, we seem to suffer from a lack of in-depth understanding of the CS framework and an inability to identify killer applications.

An area where an interest is rapidly mounting is the design of radiation detector systems. Since a detector senses radiation coming from an unknown location with unknown background and being often sparse and noisy, the signals resulting from interactions of incident radiation with the detector volume may provide an opportunity to apply the CS theory. The coded aperture imaging method, for example, has been proposed and is expected to find applications in areas such as nuclear medicine (PET and SPECT imaging), astronomy, and homeland security involving monitoring of nuclear materials [8]. The CS framework also enables the design of sub-Nyquist analog-to-digital converters (ADCs) that help eliminate the dependence on computationally intensive, high speed data acquisition and digital signal processing for compression. Various detector designs have been demonstrated for the acquisition of sparse signals in large swaths of bandwidth [9]. Ideas to modulate a photo-multiplier tube which is part of many gamma detector systems have been proposed [10]. In the near future, breakthroughs in this particular direction of research are expected to emerge by integrating CS algorithms directly with detector hardware in order to find simple and practical implementations for nuclear applications.

Monte Carlo radiation transport simulation is one of the essential tools in nuclear reactor design and analysis. In particular, methods based on solving adjoint equations, such as those implemented in production codes such as the MCNP, are widely used in shielding design, detector optimization, adjoint-weighted perturbation quantification, and even radiation treatment planning for prostate cancer [11]. The CS framework can also be useful in developing optimized algorithms for non-uniform sampling and non-uniform Fast Fourier transform (FFT) that are integral to the numerical solution of partial differential equations. CS-based acceleration by denoising and domain decomposition has been demonstrated in image processing [12]. In principle, the CS framework should be quite useful in addressing a number of reactor physics issues. An example involves the diagnosis of fission source convergence—a problem interlinking reactor physics, statistics, and acceleration methods for linear systems of equations. It may be possible that a better treatment of the Shannon entropy would lead to more accurate and rapid detection of stationarity of the fission source iteration [13]. Given the interest to accelerate

Monte Carlo simulations for global reactor analysis involving full-size reactor cores [14-16], it will not be surprising if the CS framework is found to be useful in some of these specific areas of reactor physics research.

## 5. CONCLUSIONS

The history of science and engineering shows that it is rare for a mathematical discovery to stir up so much interest from so many people in such a short period of time as CS did. Despite the broad appeal in data acquisition and processing, the viability of the highly anticipated CS framework seems to hinge on our ability to close the gap between the theory and applications in fields such as nuclear science and engineering.

## ACKNOWLEDGMENT

Mr. Mille is supported by a U.S. Department of Energy Nuclear Energy University Program Graduate Fellowship.

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