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PERTURBATION THEORY IN NUCLEAR REACTIONS

par

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Summary :

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The average amplitude is then expressed in terms of a perturbation expansion. In perturbation theory of the first kind, expansion in powers of the nucleus is carried out. In the second kind of perturbation theory, all particles are treated symmetrically and all but the average effects of the interactions are treated as perturbations. This allows one to relate the parameters of the optical potential back to nucleon-nucleon forces.

It is shown that these expansions are, in general, convergent, due to the fact that the excitation into which a given excitation decays has a longer life-time than the original one.

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Perturbation Theory in Nuclear Reactions

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Exact expressions for the amplitudes for scattering of a particle by a complex nucleus are written down. It is then shown that, with a particular weight function, the scattering amplitude can be averaged over energy by going to a complex energy, i.e., $[S(E)]_{\mathrm{Av}} = S(E + iI)$, where I is the interval averaged over.

The average amplitude is then expressed in terms of a perturbation expansion. In perturbation theory of the first kind, expansion in powers of the interaction potential between the incident particle and the particles in the nucleus is carried out. In the second kind of perturbation theory, all particles are treated symmetrically and all but the average effects of the interactions are treated as perturbations. This allows one to relate the parameters of the optical potential back to nucleon-nucleon forces.

It is shown that these expansions are, in general, convergent, due to the fact that the excitation into which a given excitation decays has a longer lifetime than the original one.

I. INTRODUCTION

In the past ten years a number of phenomena have been observed which appear to have no natural explanation in the statistical theory of nuclear reactions. Among these are the giant resonances in average cross sections for the scattering of neutrons from complex nuclei, the so-called direct interaction in inelastic scattering, giving rise to a relatively large number of fast particles, and

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the giant dipole photoeffect. It has been shown in several papers (*1–5*), in which nuclear dispersion theory has been employed, that these phenomena result from phase relations between the levels of the compound nucleus. That is, the many levels do not scatter independently, as assumed in the statistical theory, but their amplitudes combine to give an average amplitude which is responsible for the above processes. The "statistical" processes are also included in the treatment, in the part of the scattering amplitude in which phase relations are not important. The physical interpretation of the theory for elastic scattering is just that of Feshbach *et al.* (*6*).

In this paper we shall make the connection of the optical model parameters and transition amplitudes in direct interaction back to nucleon-nucleon forces. The essential point is that the average, rather than the actual amplitude is mainly responsible for the above processes, and the larger the energy region over which the average is taken, the less stringent are the criteria for the satisfaction of perturbation theory.

In Section II we formulate the problem in dispersion theory. In Section III we justify a picture developed by Lane *et al.* (*7*) which is necessary for our considerations in Section IV and Section V of the criteria for validity of perturbation theory. In Section VI we discuss the relation of our results to those of other workers, especially those of Brueckner, *et al.* (*8*) and Brueckner (*9*).

II. DEVELOPMENT

We wish to describe the elastic scattering of a single particle, the coordinate of which we denote by $\mathbf{r}$, by the initial nucleus of A particles, the totality of coordinates of which we denote by ξ. For $r < R$, the $A + 1$ particles can be described by the complete set of compound states $\Phi^{(p)}(\mathbf{r}, \xi)$, where

$$\begin{aligned} H\Phi^{(p)}(\mathbf{r}, \xi) &= W_p\Phi^{(p)}(\mathbf{r}, \xi), \\ H &= H_\xi + T(r) + V(\mathbf{r}, \xi). \end{aligned} \tag{1}$$

Here H_ξ is the Hamiltonian of the A particles, $V(\mathbf{r}, \xi)$ is the potential interaction between these A particles and the incident particle, and $T(r)$ is the kinetic energy of the latter. The W_p are the complex eigenvalues of the Kapur-Peierls formalism (*10*),

$$W_p = \epsilon_p - \frac{i\alpha_p}{2} \tag{1.1}$$

with ϵ_p and α_p real. As in Ref. *1*, we will find it convenient to introduce single-particle eigenfunctions $\hat{\psi}_m(\mathbf{r})$ in a complex well $\hat{V}(r)$ which will later be connected with the optical potential of Feshbach *et al.* (*6*). For the moment we will

consider $\hat{V}(r)$ to be a square well, $\hat{V} = -U - iW$ for $r < R$; the generalization to the case in which it varies with r is easy. We have

$$\{T(r) + \hat{V}(r)\}\hat{\psi}_m(\mathbf{r}) = E_m\hat{\psi}_m(\mathbf{r}), \tag{2}$$

where T represents the kinetic energy and

$$E_m = \epsilon_m - iW - \frac{i\beta_m}{2}. \tag{2.1}$$

The functions $\Phi^{(p)}$ and $\hat{\psi}_m$ are discussed in detail if Ref. *1*; the above definitions are given here to make this treatment reasonably self-contained.

Introducing a complete set of states $\chi_j(\xi)$ for the A particles, we can expand

$$\Phi^{(p)}(\mathbf{r}, \xi) = \sum a_{jm}{}^p\chi_j(\xi)\hat{\psi}_m{}^j(\mathbf{r}). \tag{3}$$

We make the approximations: (1) We neglect the weak dependence of $\hat{\psi}_m{}^j$ on j which arises because $\hat{V}(r)$ will turn out to be a function of the excited state j. (2) We do not antisymmetrize between the r-particle and the ξ-particles, for the moment. Whereas the identity of this last particle is certainly important in determining the numerical value of $\hat{V}$ [see, for example, the correction to the single-particle energy noted by Hugenholtz and Van Hove (*11*)], it does not seem necessary for understanding the questions we deal with here. The question of antisymmetrization will be brought up in Section V. We will further assume that $V(\mathbf{r}, \xi)$ is a well-behaved potential (i.e., a potential without a strong repulsive core and other singularities), although generalization to the case that it is not can probably be made using techniques developed by Watson and Brueckner.

The difference S^{ce} between the actual scattering amplitude and that in the complex single-particle well is the amplitude for the so-called compound elastic scattering. It is given by

$$\begin{aligned} S_\alpha{}^{ce} &\equiv S_\alpha - \hat{S}_\alpha \\ &= \frac{e^{-2ikR}}{2}\left\{\sum_p \frac{\gamma_p}{W_p - E} - \sum_m \frac{\Gamma_m}{E_m - E}\right\}, \end{aligned} \tag{4}$$

where α is a label for all channel variables. Further,

$$\gamma_p = \frac{k\hbar^2}{M} R^2 \sum_{m,m'} a_{0m}{}^p a_{0m'}{}^p \hat{\phi}_m(R)\hat{\phi}_{m'}(R), \tag{4.1}$$

$$\Gamma_m = \frac{k\hbar^2}{M} R^2[\hat{\phi}_m(R)]^2 \tag{4.2}$$

where $\hat{\phi}_m$ is the radial part of $\hat{\psi}_m$ and it is understood that both widths γ_p and

Γ_m are for the channel α; we will not bother to carry this suffix. For low bombarding energies, γ_p and Γ_m are nearly real and $\Gamma_m \cong \beta_m$, the "natural width".

It is convenient to express S^{ce} in terms of the Green's function as has been done by Bloch (*12*). Using the notation

$$\begin{aligned} \Phi^{(p)}(\mathbf{r}, \xi) &= |\, p\rangle = \langle p\,|, \\ \hat{\psi}_m(\mathbf{r})\chi_j(\xi) &= |\, jm\rangle = \langle jm\,|, \end{aligned} \tag{5}$$

we have

$$\begin{aligned} S_\alpha^{ce} &= \frac{k\hbar^2}{2M} R^2 e^{-2ikR} \sum_{m,m'} \left\{ \langle p\,|\,0m\rangle \langle p\,|\,0m'\rangle \left\langle p \left| \frac{1}{H - E} \right| p \right\rangle \right. \\ &\quad \left. - \left\langle 0m \left| \frac{1}{\hat{H} - E} \right| 0m' \right\rangle \right\} \hat{\phi}_m(R)\hat{\phi}_{m'}(R) \\ &= \frac{k\hbar^2}{2M} R^2 e^{-2ikR} \sum_{m,m'} \left\{ \left\langle 0m \left| \frac{1}{H - E} \right| 0m' \right\rangle \right. \\ &\quad \left. - \left\langle 0m \left| \frac{1}{\hat{H} - E} \right| 0m' \right\rangle \right\} \hat{\phi}_m(R)\hat{\phi}_{m'}(R). \end{aligned} \tag{6}$$

Using the identity

$$\begin{aligned} \frac{1}{H - E} &= \frac{1}{\hat{H} - E} - \frac{1}{\hat{H} - E}(V - \hat{V}) \frac{1}{\hat{H} - E} \\ &\quad + \frac{1}{\hat{H} - E}(V - \hat{V}) \frac{1}{H - E}(V - \hat{V}) \frac{1}{\hat{H} - E}, \end{aligned} \tag{6.1}$$

we obtain

$$\begin{aligned} S_\alpha^{ce} &= -\frac{k\hbar^2}{2M} R^2 e^{-2ikR} \sum_{m,m'} \frac{1}{E_m - E} \frac{1}{E_{m'} - E} \left\{ \langle 0m\,|\, V - \hat{V}\,|\,0m'\rangle \right. \\ &\quad \left. - \langle 0m\,|\,(V - \hat{V}) \frac{1}{H - E}(V - \hat{V})\,|\,0m'\rangle \right\} \hat{\phi}_m(R)\hat{\phi}_{m'}(R). \end{aligned} \tag{7}$$

Using Eq. (19) of Ref. *1* we could easily carry out the sums to obtain the basic Eq. (14.2) of Ref. *1*,

$$S_\alpha^{ce} = -\frac{2M}{\hbar^2} \left\{ \langle 0\alpha\,|\, V - \hat{V}\,|\,0\alpha\rangle - \langle 0\alpha\,|\,(V - \hat{V}) \frac{1}{H - E}(V - \hat{V})\,|\,0\alpha\rangle \right\}, \tag{7.1}$$

where $|\,\alpha\rangle$ is the scattering state which is asymptotic to the initial wave plus an outgoing wave in channel α.

We will define $\hat{V}(r)$ by requiring that $(S_\alpha)_{\text{Av}} = \hat{S}_\alpha$, where $(S_\alpha)_{\text{Av}}$ is the scat-

tering amplitude averaged with respect to the energy.[1] With this definition, $\hat{S}_\alpha$ is just the "shape-elastic" scattering of Feshbach *et al.* We will discuss in detail later just how this average should be carried out. Our requirement is, then, that $(S_\alpha^{cc})_{\text{Av}}$ be zero. It is clear from either Eq. (7) or Eq. (7.1) that this can be accomplished by requiring

$$\langle 0 \mid V - \hat{V} \mid 0 \rangle - \langle 0 \mid (V - \hat{V}) \frac{1}{H - E} (V - \hat{V}) \mid 0 \rangle_{\text{Av}} = 0, \tag{8}$$

and this will be our defining equation for $\hat{V}$. (The first term in Eq. (8) varies slowly with energy and need not be averaged.)

We now have the necessary formalism, and will concentrate in the following on the physical picture.

III. PICTURE OF LANE, THOMAS AND WIGNER

We will now discuss the picture of Lane *et al.* (7) which will be necessary for our later development. According to these authors, for $W_p \sim \epsilon_n$, i.e., for energies of the compound state in the neighborhood of the single-particle energy ϵ_n, only the term $m = m' = n$ is important in Eq. (4.1) and the $\langle on \mid p \rangle^2$ are large only in the neighborhood of ϵ_n. Another way of stating it is that in the strength function $(\gamma_p/D)_{\text{Av}}$, only terms in the expression (4.1) with $m = m' = n$ are important for such energies, and the strength function should have a pronounced maximum in the region $E \sim \epsilon_n$. Remembering that W_p has a small negative imaginary part, we can easily obtain the strength function in the following way:

$$i\pi \left(\frac{\gamma_p}{D}\right)_{\text{Av}} = \text{Im}\left(\sum \frac{\gamma_p}{W_p - E}\right)_{\text{Av}} = \frac{k\hbar^2 R^2}{M} \text{Im} \sum_{m,m'} \left\{ \langle 0m \left| \frac{1}{H - E} \right| 0m' \rangle \hat{\phi}_m(R) \hat{\phi}_{m'}(R) \right\}_{\text{Av}}. \tag{9}$$

In computing the right hand side of Eq. (9) we can make use of the identity, Eq. (6.1) and our definition of $\hat{V}$. Then,

$$\left\langle 0m \left| \frac{1}{H - E} \right| 0m' \right\rangle_{\text{Av}} = \left\langle 0m \left| \frac{1}{\hat{H} - E} \right| 0m' \right\rangle. \tag{10}$$

[1] By the average of a function $F(E)$ we mean

$$F(E)_{\text{Av}} = \int F(E') W(E - E')\, dE',$$

where W is a weight function which we shall specify later [see Eq. (12.1)]. As far as the present considerations go, we could use the rectangular weight function employed by Feshbach *et al.*, with which

$$F(E)_{\text{Av}} = \frac{1}{I} \int_{E - I/2}^{E + I/2} F(E')\, dE'.$$

We easily obtain the simple result

$$i\pi \left(\frac{\gamma_p}{D}\right)_{\text{Av}} = \text{Im} \sum_m \frac{\Gamma_m}{E_m - E} . \tag{10.1}$$

For the single particle resonance n, which is assumed to be at low energy, Γ_n is nearly real and approximately equal to β_n of Eq. (2.1), so that the contribution from it will be

$$i\pi \left(\frac{\gamma_p}{D}\right)_{\text{Av}} \cong \frac{\Gamma_n(W + \beta_n/2)}{(\epsilon_n - E)^2 + (W + \beta_n/2)^2} , \tag{11}$$

i.e., of order Γ_n/W on the single-particle resonance (at low energies we can neglect β_n, which depends linearly on k, compared with W). It is clear that the contribution of the other levels will be at least of order $W/\Delta E_m$ smaller,[2] where ΔE_m is the distance between single-particle levels of the *same angular momentum* (i.e., same channel α). This distance is of the order of U, the real part of $\hat{V}$.

It is seen from the foregoing that the validity of the picture of Lane *et al.* depends on $W \ll U$, which is also one of the criteria for the existence of marked resonances in $\hat{V}$.

We will digress for a moment to discuss W and ΔE_m for the actual physical case, taking the parameters of $\hat{V}$ from empirical fits. Let us consider, by way of example, the nucleus $A = 160$, where the $4s$ single-particle resonance occurs at zero energy experimentally, and take $\hat{V}$ as a square well of radius R. The occurrence of the $4s$ resonance requires $KR \cong (7/2)\pi$, where K is the wave number for a particle of zero energy, measured from the bottom of the well. The $5s$ resonance will occur for $K'R \cong (9/2)\pi$. Hence $E'/E = 81/49$, where E and E' are measured from the bottom of the well. For a well depth of 42 Mev, as used by Feshbach *et al.*, $\Delta E = E' - E = 27$ Mev. Further, the Γ_m's are mainly real for low bombarding energies, so that the contribution of the $5s$ level to (10.1) is really of order $W^2/(\Delta E)^2$. The values of W are, at most, a few Mev for bombarding energies of several Mev, and in the low energy region considered by Feshbach *et al.* only 1–2 Mev. In fact, when the spin-orbit force is included in the empirical fits they will probably be even smaller [Brown and De Dominicis (*5*), p. 76]. Thus, there is little doubt of the satisfaction of the condition $W/(\Delta E_m) \ll 1$[3], which is the criterion for the validity of the model.

[2] Since the sum in Eq. (10.1) relates only to the single-particle well, one can, of course, carry out the calculation of the sum explicitly for special cases, such as a square well, and verify that if $E \cong \epsilon_n$, the sum of terms $m \neq n$ contribute in order $W/\Delta E_m$ (See the Appendix of Ref. *5*).

[3] This discussion of the contribution of terms $m \neq n$ in Eq. (10.1) has been carried out taking the same W for all terms m, contrary to the spirit of Ref. *5*. [See Appendix C of this reference where it is pointed out that $\hat{V}$ is a velocity dependent well in the formulation of Refs. *1–5*.] However, the sum of terms $m \neq n$ must be the same in either method.

IV. PERTURBATION THEORY OF THE FIRST KIND

In the perturbation theory we wish to develop, it is clear that we will have to carry out an expansion in $\delta V = V(\mathbf{r}, \xi) - \hat{V}(r)$, since we wish to obtain expressions for quantities which do not contain the complicated Φ^p. It is well known that expansion of the actual wave functions $\Phi^{(p)}$ in terms of $\hat{\psi}_m$ and χ_j converges very slowly, if at all. However, the perturbation expansions of certain average quantities, such as the average phase, involve less stringent criteria for convergence,[4] which we shall now formulate.

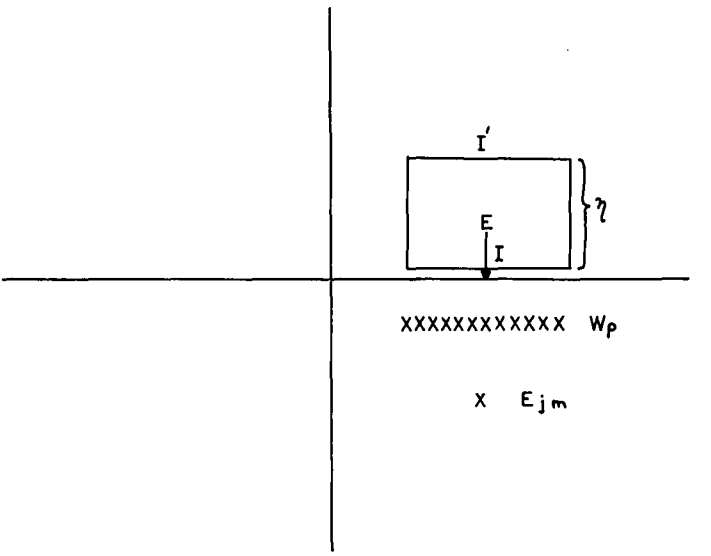

FIG. 1. Poles in the complex energy plane. The original average taken over the energy interval I can be carried out over the interval I' if the condition $\eta \ll I$ is satisfied.

In all quantities we wish to average, we will have denominators of the form $W_p - E$, where W_p has a negative imaginary part. Let us consider by way of example, the function

$$R(E) = \sum_p \frac{\gamma_p}{W_p - E}. \tag{12}$$

All of the poles of $R(E)$ will lie below the real axis in the complex energy plane (see Fig. 1) and we can use the method of Wigner (*13*) to average this function over energy. Thus, $\int R(E)\, dE = 0$ if dE follows a contour lying wholly above the

[4] A similar situation occurs in calculation of the thermodynamic partition function, which is insensitive to perturbations smaller than kT, although they may be large compared with the distances between levels.

real axis. If we can neglect end effects, then the integral of $R(E)$ along the strip I is equal to that along I'. That is, if η is the distance of I' above the real axis, then $R(E)_{\mathrm{Av}} = R(E + i\eta)_{\mathrm{Av}}$ neglecting end effects. Such a procedure is often convenient because addition of the imaginary part $i\eta$ removes the rapid fluctuations encountered in $R(E)$ whenever $E = \epsilon_p$.

This method gives inessential complications because of the end effects. We can avoid these by defining the average with respect to the smooth weight function

$$W(E - E') = \frac{I}{\pi} \frac{1}{(E - E')^2 + I^2} \tag{12.1}$$

instead of the rectangular weight function employed above. Here I corresponds to the interval we are averaging over. Then

$$\begin{aligned} R(E)_{\mathrm{Av}} &\equiv \frac{I}{\pi} \int_{-\infty}^{\infty} \frac{R(E')}{(E - E')^2 + I^2} \, dE' \\ &= \sum_p \frac{\gamma_p}{W_p - E - iI} = R(E + iI). \end{aligned} \tag{13}$$

Consequently, averaging with respect to this weight function is equivalent to adding a positive imaginary part, equal to the interval averaged over, to the energy.

It is clear that if such average quantities are not to fluctuate, the interval I must cover many compound states p.

By applying this method to Eq. (8), we find that the defining equation for $\hat{V}$ becomes

$$\left\langle 0 \left| (V - \bar{V} + \mathfrak{W}) \left[1 - \frac{1}{H - E - iI} (V - \bar{V} + \mathfrak{W}) \right] \right| 0 \right\rangle = 0, \tag{14}$$

where we have separated $\hat{V}$ into

$$\hat{V} = \bar{V} - \mathfrak{W}, \tag{15}$$

with

$$\bar{V} \equiv \langle 0 \mid V \mid 0 \rangle. \tag{15.1}$$

Now, Eq. (14) can be explicitly solved for $\mathfrak{W}$. For brevity, we call

$$\begin{aligned} e &= H = E - iI, \\ \bar{e} &= \bar{H} - E - iI = e - (V - \bar{V}). \end{aligned} \tag{16}$$

then Eq. (14) can be written

$$\begin{aligned}\langle 0|(V-\bar{V}+\mathfrak{W})\left[1-\frac{1}{e}(e-\bar{e}+\mathfrak{W})\right]|0\rangle \\ =\mathfrak{W}\left[1-\langle 0|(V-\bar{V})\frac{1}{e}|0\rangle\right]+\left\langle 0|(V-\bar{V})\frac{1}{e}\bar{e}|0\right\rangle=0,\end{aligned} \tag{17}$$

where we have made use of the fact that [see Eq. (10)]

$$\left\langle 0\left|\frac{1}{e}\right|0\right\rangle=\frac{1}{\bar{e}-\mathfrak{W}}. \tag{17.1}$$

The last equation in (17) gives immediately

$$\begin{aligned}\mathfrak{W}&=\frac{1}{1-\langle 0|(V-\bar{V})\frac{1}{e}|0\rangle\left\langle 0|(V-\bar{V})\frac{1}{e}(V-\bar{V})|0\right\rangle} \\ &=\left\langle 0|(V-\bar{V})\frac{1}{e-Q_0(V-\bar{V})}(V-\bar{V})|0\right\rangle,\end{aligned} \tag{18}$$

where

$$Q_0\equiv|0\rangle\langle 0|. \tag{18.1}$$

Eq. (18) has been obtained independently by Feshbach (*14*).

In working with Eq. (18) it clearly will be advantageous to work with a representation for the extra particle which is diagonal in $T+\bar{V}$ rather than in $T+\hat{V}$. For the simple case we have been considering in which $\hat{V}$ is a square well, the eigenfunctions are the same for the two cases, but the energies differ by iW. We shall therefore not bother to change the notation, but it should be remembered that the single-particle energies do not now contain W, i.e., the new E_m are $E_m=\epsilon_m-i\beta_m/2$.

We now investigate the eigenfunctions and eigenvalues of the operator

$$\mathfrak{K}=H-Q_0(V-\bar{V}).$$

We label them by Ω_p and W_p', i.e.,

$$\mathfrak{K}\Omega_p=W_p'\Omega_p. \tag{19}$$

Now $-Q_0(V-\bar{V})$ can be considered as a small perturbation since the Ω_p are not very different from the $\Phi^{(p)}$. Thus, we choose

$$\begin{aligned}\Omega_p&=\Phi^{(p)}+\delta\Omega_p, \\ \delta\Omega_p&=\sum{}'a_p{}^n\Phi^{(n)}.\end{aligned} \tag{20}$$

To lowest order

$$W_p' = W_p . \tag{20.1}$$

Using Green's theorem and assuming Ω_p to satisfy the same boundary conditions as $\Phi^{(p)}$, we obtain

$$\int_0^R (\Phi^{(n)}\mathfrak{K}\Omega_p - \Omega_p\mathfrak{K}\Phi^{(n)})d\tau = 0$$
$$= (W_p - W_n)a_p{}^n + \sum_m \langle p|(V - \bar{V})|0m\rangle\langle 0m|n\rangle, \tag{21}$$

where we have set $W_p' = W_p$ and $\Omega_p = \Phi^{(p)}$ on the right-hand side, since we wish to compute $a_p{}^n$ only to lowest order. Employing

$$\langle p \mid V - \bar{V} \mid 0m\rangle = \langle p \mid H - \bar{H} \mid 0m\rangle = \left(W_p - \epsilon_m - i\frac{\beta_m}{2}\right)\langle p \mid 0m\rangle, \tag{22}$$

we obtain

$$a_p{}^n = \sum_{n,m} \frac{\left(W_p - \epsilon_m - i\frac{\beta_m}{2}\right)}{W_n - W_p}\langle p \mid 0m\rangle\langle 0m \mid n\rangle. \tag{23}$$

Now, for order of magnitude estimates,

$$\langle p \mid 0m\rangle\langle 0m \mid n\rangle\beta_m \sim \langle p \mid 0m\rangle^2\beta_m \sim \gamma_p$$

and, on the giant resonance,

$$W_p - \epsilon_m \sim W.$$

It is clear, therefore, that the $a_p{}^n$ are large only if $W_p - W_n \sim W\gamma_p/\beta_m$, and, therefore, because of the smallness of γ_p, only a few of the neighboring Φ's are mixed into each Ω by the perturbation.

Consequently, we see from Eqs. (20.1) and (23) that the term $-Q_0(V - \bar{V})$ viewed as a perturbation, does not change the eigenvalues appreciably and that the new eigenfunctions are combinations of the old ones taken over a narrow range.

We are now ready to discuss the criteria for perturbation theory. We can express the matrix element as

$$\langle 0m \mid \mathfrak{W}(E) \mid 0m\rangle = \langle 0m \mid V - \bar{V} \mid p\rangle \frac{1}{W_p - E - iI}\langle p \mid V - \bar{V} \mid 0m\rangle. \tag{24}$$

In a perturbation expansion of the type

$$\mathfrak{W}(E) = \langle 0 \mid (V - \bar{V}) \frac{1}{\bar{H} - E - iI} \sum_{n=0}^{\infty} (-1)^n \cdot \left((1 - Q_0)(V - \bar{V}) \frac{1}{\bar{H} - E - iI} \right)^n (V - \bar{V}) \mid 0 \rangle, \tag{25}$$

the lowest order term of the expansion of (25) can be expressed as

$$\langle 0m \mid \mathfrak{W}(E) \mid 0m \rangle \cong \sum_{j,n} \frac{\langle 0m \mid V - \bar{V} \mid jn \rangle \langle jn \mid V - \bar{V} \mid 0m \rangle}{E_{jn} - E - iI}. \tag{26}$$

The same result can be obtained by replacing $\mid p \rangle$ by $\sum_{j,n} \mid jn \rangle \langle jn \mid p \rangle$ in Eq. (24) and then approximating W_p by E_{jn}. This approximation is justified if I is large, because the $\langle jn \mid p \rangle$ are large only if $W_p - \epsilon_{jn} \sim W$ by the arguments of the preceding section. We illustrate the situation in Fig. 2. In the complex energy plane the state $\mid jn \rangle$ will have an appreciable probability of being found in states $\mid p \rangle$, as shown above, distributed over a region of width W. It can be seen that what we are doing in going from Eq. (24) to Eq. (26) is approximating the distances in the energy plane from $E + iI - W_p$ by the common distance $E + iI -$

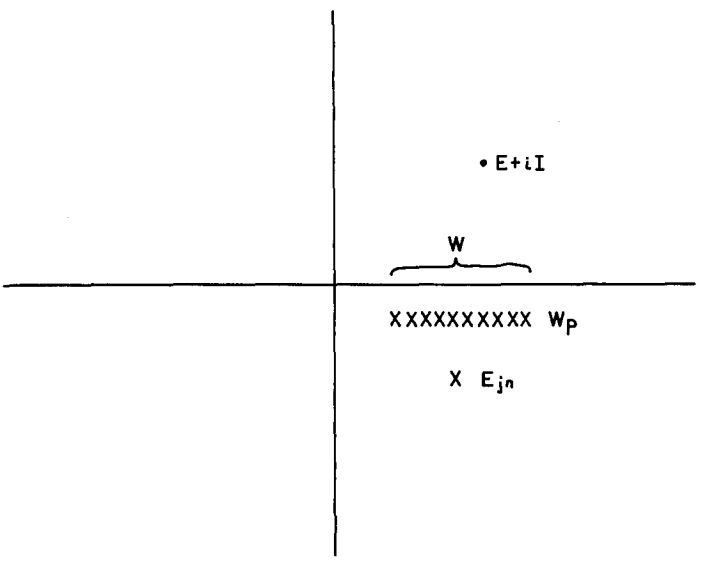

FIG. 2. Poles in the complex energy plane. With the weight function of Eq. (12.1) the average of a function of E over the interval I is given by the value of the function at $E + iI$.

E_{jn}. If $E - \epsilon_{jn} \gg W$, this is justified. This case was discussed in detail in Ref. *1*. We show in the figure the worse case that can happen for a given I, i.e., where $\epsilon_{jn} \cong E$. Then, clearly we must have $I > W$ for the expansion to converge.

The expansion of $\mathfrak{W}$ in the above fashion has been carried out by Bloch (*12*). With assumptions of a randomness in signs of the matrix elements he arrives at the conclusion that Eq. (26) is a good approximation to Eq. (24). We do not, however, want to make such an assumption, and we will see later that it is, in fact, unjustified.

Now, the decisive point is that W is a function of E.[5] The total width of the state $|\, jn\rangle$ is determined primarily by the value of W for a particle of energy $E - \epsilon_j$ above the Fermi energy. (We neglect the width of the excited state $\chi_j(\xi)$; since it is a true compound state, its width is much smaller than that of the single particle state $\hat{\psi}_m$ in the well). Taking the quadratic dependence of W given by simple theories (*8, 15, 16*) which we will discuss later,

$$W_j = \left(\frac{E - \epsilon_j}{E}\right)^2 W, \tag{27}$$

where we have put a lower suffix j on W to indicate that it refers to the state jn. In other words, the width of the state that the single particle excitation decays into is less than the width of the original single-particle excitation.

Usually, the state j will be a highly excited one, since the number of states per Mev available increases exponentially with excitation,[6] and, in this case, $W_j \ll W$. Of course, the more highly excited states tend also to have a more complicated structure, so that the matrix elements for excitation of the A particles become smaller. We shall see in the next section that the ratio $E - \epsilon_j$ tends to be $\sim 1/3$ or, $W_j/W \sim 1/9$. However, in the case of easily deformable nuclei, there are strong matrix elements to low-lying states j, the collective ones. We will return to consideration of these nuclei later.

From the foregoing, we see that we can usually satisfy the inequalities

$$W_j < I < W. \tag{28}$$

Consequently, we can average over an interval which is large enough for the convergence of perturbation theory, but small enough so that one can still obtain information about the shape of the single-particle resonance, of width $\sim W$. (In fact, our neglect of variation in the $|\, m\rangle$ in the various averages taken, constrain us to average over a distance $< W$). We have, therefore, given a justification of

[5] That this is important was pointed out by Prof. Bethe (private communication).

[6] The energy of excitation is limited, since we must assume, for reasonability, that the initially excited particle does not drop into a state already occupied. This is inconsistent with our neglect of antisymmetrization, but this does not seem to be an essential difficulty.

the expansion introduced by Bloch (*12*) (but not of his random phase approximation, which we shall return to in Section VI).

With slight modification, the same arguments apply to the case of inelastic processes, going into a low excited final state. Here the transition amplitude is

$$S^{in}_{\alpha,\alpha' j} = -\frac{2M}{\hbar^2}\left\{\langle j\alpha' \mid V - \hat{V} \mid 0\alpha\rangle - \langle j\alpha' \mid (V - \hat{V})\frac{1}{H - E}(V - \hat{V})\mid 0\alpha\rangle\right\}$$

$$= -\frac{k\hbar^2}{2M} R^2 e^{-2ikR} \sum_{m,m'} \frac{1}{E_m - E}\frac{1}{E_{m'} - E}\left\{\langle jm' \mid V - \hat{V} \mid 0m\rangle \right. \tag{29}$$

$$\left. - \langle jm' \mid (V - \hat{V})\frac{1}{H - E}(V - \hat{V}) \mid 0m\rangle\right\},$$

where j is assumed to be a low excited state. Perturbation theory can be used to calculate $(S^{in}_{\alpha,\alpha' j})_{\mathrm{Av}}$ since the states that m and m' decay into are, in general, of substantially lower energy and, consequently, have a smaller width. This gives the result, in first approximation,

$$(S^{in}_{\alpha,\alpha' j})_{\mathrm{Av}} \cong -\frac{2M}{\hbar^2}\langle j\alpha' \mid V - \bar{V} \mid 0\alpha\rangle, \tag{29.1}$$

which has been used extensively in calculating direct interaction processes [see, e.g., Levinson and Banerjee (*17*)]. Thus, our justification of perturbation theory of the first kind provides a justification for this type of calculation.

V. PERTURBATION THEORY OF THE SECOND KIND

The relation (29.1) is an important one, in that it justifies the use of perturbation theory—in the usual terminology, Born approximation with distorted waves —in direct interaction calculations. We found, also, in the last section, that the average elastic scattering phase was reproduced by $\hat{V} = \bar{V} + \mathfrak{W}$ with $\mathfrak{W}$ given to lowest order by Eq. (26). This latter relation is interesting, but not well adapted for the calculation of $\mathfrak{W}$, since it still involves the highly complicated nuclear states $\chi_j(\xi)$. We therefore develop a perturbation expansion of the type used by Brueckner *et al.* (*8*), which essentially relates the nuclear states $\chi_j(\xi)$ back to shell model states. This expansion gives an expression for $\mathfrak{W}$ which is practical for calculation; however, to satisfy the criterion for its validity, one must average over a large energy interval, as we shall see.

Before proceeding with the general development, we shall first sketch an extension of the theory of the preceding section, which is helpful in understanding the later developments. We introduce the two component excitation

$$\mid jmn\rangle = \Lambda_j(\xi, \mathbf{r}_k^{-1})\hat{\psi}_m(\mathbf{r}_0)\hat{\psi}_n(\mathbf{r}_k), \tag{30}$$

where the Λ_j are the eigenfunctions of the A-1 particles in the nucleus after removal of the kth particle, and $\hat{\psi}_m$ and $\hat{\psi}_n$ are the eigenfunctions of the r_0-particle (which we denoted with the r-coordinate before) and the r_k-particle in the well $\hat{V}'$, produced by the A-1 particles. We can again expand

$$\Phi^{(p)}(\mathbf{r}_0, \xi - 1, \mathbf{r}_k) = \sum_{jmn} b^p_{jmn} \Lambda_j \hat{\psi}_m \hat{\psi}_n . \tag{31}$$

Most of the development is, by now, fairly obvious, and we quote only the results. Perturbation theory involves forming the unit operator $\sum_{j,k,l} |jkl\rangle\langle jkl|$ and then approximating the denominator in Eq. (24) by $E_{jkl} - E - iI$ when the jkl term occurs in the unit operator. From the development of the last section, we see that this replacement is justified if I is greater than the range over which the $\langle jkl|$ is spread into the $\langle p|$. This spread comes from three sources.

(i) and (ii). The particle in either state k or l can interact with one of the $A - 1$ particles in state $\langle j|$ so that the two-particle excitation, in perturbation theory language, decays into a three-particle excitation. To order $1/A$, the width from these processes is just the sum of the imaginary parts of the single-particle energies E_k and E_l in the original well $\hat{V}(r)$.

(iii) The two particles in state k and l can interact and the two-particle excitation can go back into a single-particle excitation $|on\rangle$. The width from this is small compared with the width of the single-particle excitation $|om\rangle$ which can decay into a large number of different two-particle excitations with similar matrix elements. Only this contribution to the two-particle width was noted in Brueckner *et al.* (*8*), and, consequently, these authors obtained much too low an estimate of the width of the two-particle excitation.

The contribution from (i) and (ii) can be estimated in the same way as we arrived at Eq. (27). Thus, in terms of the imaginary part of $\hat{V}$ for the initial single-particle excitation (which we now label by W_m),

$$W_k + W_l = \left\{ \frac{(\epsilon_k - E_F)^2 + (\epsilon_l - E_F)^2}{(\epsilon_m - E_F)^2} \right\} W_m . \tag{32}$$

Since $\epsilon_k - E_F$ and $\epsilon_l - E_F$ are small compared with $\epsilon_m - E_F$, $W_k + W_l$ will be less than W_m. In most cases, the single-particle excitations will decay into a state in which both particles are far down in the well, so that the width of the two-particle excitation will be substantially less than that of the original state. Then we can satisfy $W_k + W_l < I < W_m$. The first inequality is the criterion for perturbation theory of the type employed, and the second one is necessary if any information about the detailed shape of the giant resonance is to be obtained and if our averaging procedure is to be consistent. (We neglected the variation in $|m\rangle$ over the interval I.)

There is, however, another width inherent in this treatment—at least in a practical sense—and that arises because we don't know the E_j. In so far as there is

a spread in the ϵ_j for states Λ_j which contribute to the sum $\sum_{j,k,l} | jkl> < jkl |$, we must average over an energy interval large enough so that this further spread is not important. The origin of this spread can be understood if we consider the original state $\chi_0(\xi)$ to be a shell-model state. Then, the state formed by removing the kth particle, i.e., the shell-model state with a hole in it, will not be an eigenstate of the Hamiltonian for the remaining A-1 particles, but the probability of finding it in one of the eigenstates j of energy near that of the shell model state with a hole in it will be unity. The width connected with this is related to the inverse of the time in which the shell-model state with a hole into it decays into eigenstates of the residual system [see Wilkinson (*18*), p. 1053, where a discussion of just this effect, relative to the giant dipole photoeffect, is given].

Although the physical picture is simple in this formalism, it is not easy to take into account the identity of the particles by antisymmetrization since the A-1 particles are described by one Hamiltonian, and the kth particle by another. We now introduce a general formalism that treats them symmetrically, and also allows us to calculate the width for the dissolving of the shell-model state with a hole in it.

We will employ, as unit operator in Eq. (24), $| \bar{q}\rangle \langle \bar{q} |$ where $| \bar{q}\rangle$ are the eigenstates of

$$\bar{H} = \sum_{i=0}^{A} (T_i + V_i) - \frac{1}{2} \sum_{i,j} \bar{V}_{ij}, \tag{33}$$

where V_i is the self-consistent potential defined in the state $| \bar{q}\rangle$ felt by the ith particle, and

$$\bar{V}_{ij} = \langle \bar{q} | V_{ij} | \bar{q}\rangle. \tag{34}$$

The c-number $\frac{1}{2} \sum_{i,j} \bar{V}_{ij}$ has been subtracted to ensure that $\langle \bar{q} | \bar{H} | \bar{q}\rangle$ is equal to the energy in the Hartree-Fock approximation. The crucial point in using the $| \bar{q}\rangle$ as zero-order functions is then as to the extent of the spread in energy of the strength function $\langle \bar{q} | p\rangle^2$, which can be obtained from

$$\left\{\frac{\pi \langle \bar{q} | p\rangle^2}{D}\right\}_{\text{Av}} = \operatorname{Im} \left\langle \bar{q} \middle| \frac{1}{H - E} \middle| \bar{q} \right\rangle_{\text{Av}} = \operatorname{Im} \left\langle \bar{q} \middle| \frac{1}{\hat{H} - E} \middle| \bar{q} \right\rangle, \tag{35}$$

where $\hat{H}$ has been introduced as an artifice to calculate this strength function (its significance will be seen later) and it is given by

$$\hat{H} = \bar{H} - \mathfrak{W}, \tag{35.1}$$

where $\mathfrak{W}$ is then defined by

$$0 = \langle \bar{q} | \mathfrak{W} | \bar{q}\rangle - \langle \bar{q} | (V - \bar{V} + \mathfrak{W}) \frac{1}{H - E - iI} (V - \bar{V} + \mathfrak{W}) | \bar{q}\rangle. \tag{36}$$

[See the similar development, Eqs. (6–8).] This equation is formally similar to Eq. (17) and by the same procedure as employed to solve it, we can find that it can be satisfied by choosing $\mathfrak{W}$ to be diagonal in the $|\bar{q}\rangle$ representation, with matrix elements

$$\langle\bar{q}|\mathfrak{W}|\bar{q}\rangle = \langle\bar{q}|(H-\bar{H})\frac{1}{H-Q_{\bar{q}}(V-\bar{V})-E-iI}(H-\bar{H})|\bar{q}\rangle, \tag{37}$$

where

$$Q_{\bar{q}} \equiv |\bar{q}\rangle\langle\bar{q}|. \tag{37.1}$$

In evaluating $\langle\bar{q}|\mathfrak{W}|\bar{q}\rangle$ by lowest order perturbation theory, which now means replacing

$$\{H-Q_{\bar{q}}(H-\bar{H})-E-iI\}^{-1}$$

by

$$|\bar{q}'\rangle\{E_{\bar{q}'}-E-iI\}^{-1}\langle\bar{q}'|,$$

we find that

$$\mathfrak{W} = \sum_{i<j}\mathfrak{W}_{ij}$$

with

$$\langle\bar{q}|\mathfrak{W}_{ij}|\bar{q}\rangle = \sum_{\bar{q}'\neq\bar{q}}\langle\bar{q}|V_{ij}|\bar{q}'\rangle\frac{1}{E_{\bar{q}'}-E-iI}\langle\bar{q}'|V_{ij}|\bar{q}\rangle. \tag{38}$$

he half-width of the strength function $\pi\{\langle\bar{q}|p\rangle^2/D\}_{\mathrm{Av}}$ is given by

$$\sum_{i<j}^{A}\mathrm{Im}\,\langle\bar{q}|\mathfrak{W}_{ij}|\bar{q}\rangle$$

following Eq. (35).

If we identify $|\bar{q}\rangle$ with the state in which A-1 particles form a state with the A levels from the bottom of the well filled except for one hole in state k', and in which 2 particles are excited, then it is easy to identify the various parts of the width from the previous discussion:

$$\mathrm{Im}\sum_{j=1}^{A}\langle\bar{q}|\mathfrak{W}_{kj}|\bar{q}\rangle$$

and

$$\mathrm{Im}\sum_{i=1}^{A}\langle\bar{q}|\mathfrak{W}_{0i}|\bar{q}\rangle$$

correspond to the widths listed under (i) and (ii); Im $\langle \bar{q} \mid \mathcal{W}_{k0} \mid \bar{q} \rangle$ corresponds to that listed under (iii), and can generally be neglected compared with the others. The sum Im $\sum'_j \langle \bar{q} \mid \mathcal{W}_{k'j} \mid \bar{q} \rangle$ gives the width for the decay of the shell-model state with a hole in it, i.e., the width for absorption of the hole. Using the approximate symmetry between holes and particles near the Fermi surface, we find that the total width of $\mid \bar{q} \rangle$ is

$$W_{\bar{q}} = W_k + W_l + W_{k'} = \left\{ \frac{(\epsilon_k - E_F)^2 + (\epsilon_l - E_F)^2 + (E_F - \epsilon_{k'})^2}{(\epsilon_m - E_F)^2} \right\} W_m \tag{39}$$

with

$$\epsilon_k + \epsilon_l + \mid \epsilon_{k'} \mid \cong \epsilon_m .$$

Clearly, the largest phase space is available when the energy is equally distributed among the particles and the hole, so that W_k, W_l and $W_{k'}$ are each $\sim(1/9)W_m$. Since $W_k + W_l + W_{k'} < W_m$, we can again satisfy

$$W_{\bar{q}} < I < W_m \tag{40}$$

and employ perturbation theory, although it is clear that all three of these quantities are of the same order of magnitude, so that the average must be carried out over an interval I that is a good fraction of the width of the single-particle resonance.

In case $\mid 0 \rangle$ can be described as a shell-model state, i.e., a state of independent particles with the ith particle moving in a well

$$\sum_{j=0}^{A}{}' \langle 0 \mid V_{ij} \mid 0 \rangle$$

it is clear that the identity of the particles can easily be taken into account. In this case, we can assign an imaginary part to the energy of the incident particle of

$$W_m = \text{Im} \frac{< 0m \mid \left(\sum_{i=1}^{A} V_{0i} - \sum \bar{V}_{0i} \right) \mid \bar{q} \rangle \langle \bar{q} \mid \left(\sum V_{0i} - \sum \bar{V}_{0i} \right) \mid 0m \rangle}{E_{\bar{q}} - E - iI} \tag{41}$$

with both $\mid om \rangle$ and $\mid \bar{q} \rangle$ completely antisymmetrical states. In the case of many states $\mid \bar{q} \rangle$, this will be approximately

$$W_m = \frac{\pi}{D} \langle 0m \mid \left(\sum V_{0i} - \sum \bar{V}_{0i} \right) \mid \bar{q} \rangle^2_{\text{Av}} , \tag{42}$$

and it is just this quantity that has been evaluated for the case of the infinite nucleus by Brueckner *et al.* (*8*)

VI. DISCUSSION

The main function of the development in this paper is to make the connection between average properties of the many-body system and those calculated in the simple picture of particles in a complex well. We defined single-particle and two-particle excitations in this complex well, and showed how the widths of these excitations were, on the one hand, related to the distance over which these simple excitations had an appreciable probability of occurring in the true compound states of the system of $A + 1$ particles, and, on the other hand, related to the imaginary part of the complex potential. In other words, we related the relevant strength functions to parameters of the complex potential.

It was shown that the width of the two-particle excitation is determined mainly by the sum of probabilities of either of the two excited particles or the hole going into compound states [see the discussion following Eq. (38)], and that the probability of the two particles interacting so that the two-particle excitation returns to a single-particle one—i.e., so that one of the particles drops into the hole—is very small compared with this. In the work of Brueckner *et al.* (*8*), only this latter effect was taken into account, and they consequently obtained much too small a value for the width of the two-particle excitation.[7]

The fact that the widths of the important two-particle excitations are of the order of Mev means that one has to average over large energy intervals to satisfy the criterion for what we termed perturbation theory of the second kind, which was used by Brueckner *et al.* We showed, however, by assuming W to have a simple quadratic dependence on energy, as given by the lowest-order expression in perturbation theory, that the two-particle excitation is smaller, by a numerical factor, than the single-particle one, and that, therefore, one could both satisfy the criterion for perturbation theory and average over an energy interval smaller than the width of the single-particle excitation. The three-particle excitation tends, by the same argument, to be smaller than the two-particle one, etc., so that good numerical results could be expected, by these arguments, if one goes to higher orders.

In a paper following Ref. *8*, Brueckner has shown (*9*) that iterating the simple shell-model type wave function considered in Ref. *8* increases the calculated value of W by a considerable factor, and introduces both a term linear in energy and a constant term in the dependence of W. Whereas his calculation brings out the sensitivity of W to correlations in the ground-state wave function near the Fermi surface, there are indications that these cannot be calculated in perturbation theory, i.e., that the ground state wave functions $\chi_0(\xi)$ cannot be obtained in the region of the Fermi surface by iterating the shell-model state (See Bohr *et al.* (*19*)]. Thus, we will not use Brueckner's numerical results, although we should note that terms in W linear and constant in the energy will make it more

[7] This was pointed out to one of the authors by Prof. Bethe.

difficult to satisfy our criterion for the validity of perturbation theory. It seems that this problem of correlation in the ground state wave function is important for an adequate evaluation of W.

The detailed establishment of the connection between the optical model parameters and the nucleon-nucleon force, discussed here, is by no means necessary to explain the empirical agreement observed to date at low energies between the experiments and calculations from complex wells. As discussed in Refs. *1–5* this depends only on the imaginary part of the well being small compared with the real part. The analysis of Section V showed that the optical model can be established in detail, only if one averages the phase over an interval which is a good fraction of the width of the giant resonance, and that only in this case can the connection between W and the nucleon-nucleon forces be made. It would be of interest, therefore, to see if experiments in which the averaging is carried out over smaller intervals, give results in detailed agreement with the theoretical predictions—more specifically, to see if the experimental results follow the Lorentz form of the cross section over the giant resonance, as discussed in Ref. *5*.

We note, further, that in no order of perturbation theory is one justified in neglecting phase relations, i.e., in assuming random signs for matrix elements, as has been done by many authors. If averages are carried out over energy intervals of width sufficiently large for perturbation theory to be valid, then the various integrals can be related to those involving wave functions for simple types of excitations. For large nuclei, these wave functions can be replaced by those for a Fermi gas, and the relevant matrix elements become the Fourier transforms of the nucleon-nucleon potential. These vary slowly and regularly with momentum transfer. Thus, the third order term in the perturbation expansion is nonzero, and may be of the same order as the second-order term. In fact, in cases where the second-order term is abnormally small due to the limiting of phase space by the exclusion principle, it may be even larger.

Our expressions (41) and (42) are valid approximations even in the presence of shell structure, their validity depending only on the fact that—because of the energy dependence of the imaginary part of the potential—the original single-particle excitation has a shorter lifetime (larger width) than the two-particle excitation in which the original energy is divided between the two particles. Thus, for light nuclei where there are relatively few two-particle excitations, detailed solution of Eq. (41) might give a W which varies nonmonotonically across a single-particle resonance. In going over to a Fermi gas to evaluate W, we effectively drop structure coming from shell effects.

In the discussion following Eq. (29) we promised to return to the question of easily deformable nuclei. In this case, there are large matrix elements from χ_0 to low-lying rotational states, so that the initial single-particle excitation has a high probability of decaying into an excitation where the initial particle has only

slightly less energy. This state, therefore, has a width almost as large as the initial width. In this case, one should separate out the transitions to low-lying collective states, which, together with the initial channel, we will term "chosen channels," and treat them separately in a system of coupled equations as has been done by Sano *et al.* (*20*). (References to earlier work by Yoshida and others are given in this article.)

In fact, our development, Eqs. (17) and (18) can be easily generalized so that $(V - \tilde{V})$ is a matrix between chosen channels, $|0\rangle$ becomes the space of the chosen channels, and Q_0 excludes these from occurring in intermediate states. This is within the spirit of the optical model, where the distortion provided by the optical potential is supposed to represent the average effects from the great number of channels and corresponding large number of degrees of freedom which cannot be conveniently treated in detail.

We note finally that in our discussion, we have assumed throughout that the nucleon-nucleon potentials were well-behaved. The generalization to the case where hard cores are present can probably be made using techniques developed by Watson and Brueckner, but we do not believe this essential to an understanding of the points discussed here.

After completion of this work, we received a private communication from Dr. A. Sugie which indicates that the formalism of Section IV can easily be antisymmetrized. We also received a preprint from Prof. H. Feshbach in which both the question of antisummetrization and the treatment of potentials which are not well behaved are treated, using a representation formally similar to that employed here in Section IV. Finally, A. M. Lane and J. E. Lynn discuss the problem of antisymmetrization in a preprint, "The Theory of Radiative Capture Reactions." We would like to thank all of these authors for communicating their results to us before publication.

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