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Stochastic Quantization and 1/N Expansion *

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Stochastic Quantization and $1/N$ Expansion *

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Abstract

We study the $1/N$ expansion of field theories in the stochastic quantization method of Parisi and Wu using the supersymmetric functional approach. This formulation provides a systematic procedure to implement the $1/N$ expansion which resembles the ones used in the equilibrium. The $1/N$ perturbation theory for the non linear sigma model in two dimensions is worked out as an example.

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1. Introduction

In a previous paper[1] (hereafter referred to as I) we have formulated the $1/N$ expansion[2] in the stochastic quantization method (SQM) based on the Langevin and on the functional integral approach. We studied specifically the $1/N$ expansion of the non linear sigma model (NLSM) showing its renormalizability in the functional framework. In this paper we will re-examine the $1/N$ expansion through the supersymmetric functional formulation for stochastic process where it can be implemented more naturally and systematically. In fact the same routines of the equilibrium can be applied in the present situation. This observation is true even for the renormalizability of the theory. In I the cancellation of a series of non-Lagrangian counterterms for the NLSM was achieved by explicitly use of the BRS symmetry[3] for stochastic processes as well additional Ward identities related to the constraint of the theory. In a supersymmetric perturbative treatment, as we will consider here, these cancellations are automatic and are taken into account by the supergraphs of the theory. To establish the renormalizability of the NLSM we will have only to generalize Aref'eva's identity[4] to the stochastic case.

The aim of this paper is to introduce the $1/N$ expansion of field theories in the SQM using the superfield perturbation theory throughout. The organization of the paper is the following. In Sec. 2 we review the supersymmetric formulation of the SQM stressing the important points and also introducing our notation for the rest of the paper. In Sec. 3 we carry out the $1/N$ expansion of the NLSM in two dimensions on the basis of the superfield technique. We also make the connection with the equilibrium limit. In Sec. 4 the ultraviolet behaviour of the supergraphs and the renormalizability of the theory are discussed. We end the work in Sec. 5 with our conclusions. In the appendix a derivation of the fluctuation dissipation theorem used in Sec. 3 is included for completeness.

2. Stochastic Quantization and its Supersymmetric Formulation

In the SQM devised by Parisi and Wu[5] (for a review see Ref. 6) an additional coordinate t is incorporated in the fields, $\phi(x) \to \phi(x,t)$, which in their turn obeys a dynamics given by a stochastic differential equation (Langevin equation)

$$\frac{\partial\phi(x,t)}{\partial t} = -\frac{\delta S[\phi]}{\delta\phi(x,t)} + \eta(x,t) \tag{2.1}$$

where S is the Euclidian action and $\eta(x,t)$ is a Gaussian white noise with correlations

$$\begin{aligned} &\langle\eta(x,t)\rangle = 0 \\ &\langle\eta(x_1,t_1)\,\eta(x_2,t_2)\rangle = 2\,\delta^n(x_1-x_2)\,\delta(t_1-t_2) \end{aligned} \tag{2.2}$$

The basic recipe of the SQM is to obtain the solution $\phi_\eta(x,t)$ of (2.1) subject to an initial condition, $\phi_0 = \phi(x,t=t_0)$. It is then possible to demonstrate that the Green functions of the quantum theory are given by the equilibrium correlations functions

$$\langle 0|T\phi(x_1)\phi(x_2)\cdots\phi(x_n)|0\rangle = \lim_{t\to\infty}\langle\phi_\eta(x_1,t)\phi_\eta(x_2,t)\cdots\phi_\eta(x_n,t)\rangle \tag{2.3}$$

Also, the correlation functions for $\phi_\eta(x,t)$ can be calculated through a functional generator $Z[J]$ expressed in terms of a functional integral[7]

$$Z[J] = \int \mathcal{D}\phi\,\mathcal{D}\lambda\,\mathcal{D}\overline{C}\,\mathcal{D}C\,\exp\left\{-S(\phi,\lambda,\overline{C},C) + \int d^n x dt\, J(x,t)\,\phi(x,t)\right\} \tag{2.4}$$

where

$$S = \int d^n x dt \left\{-\lambda^2(x,t) + \lambda(x,t)\left[\dot\phi(x,t) + \frac{\delta S[\phi]}{\delta\phi(x,t)}\right] - \overline{C}(x,t)\left[\frac{\partial}{\partial t} + \frac{\delta^2 S[\phi]}{\delta\phi(x,t)^2}\right]C(x,t)\right\} \tag{2.5}$$

This action has two symmetries. The first is a nilpotent one[5]

$$\begin{aligned} \delta\phi(x,t) &= \bar\epsilon C(x,t) \\ \delta C(x,t) &= 0 \\ \delta\overline{C}(x,t) &= \bar\epsilon\lambda(x,t) \\ \delta\lambda(x,t) &= 0 \end{aligned} \tag{2.6}$$

where $\bar\epsilon$ is a Grassmann variable. This symmetry is similar to the BRS one found in gauge theories[8] and implies the maintenance of the structure of the action (2.5) by renormalization. It is an exact symmetry, i.e., it does not depend on the boundary conditions imposed to the fields[9] and it is true both for Markovian and non-Markovian process. For fields in (2.5) satisfying periodic boundary conditions (or anti-periodic) we have the following anti-BRS symmetry (called supersymmetry in the literature when combined with (2.6))[10−13]

$$\begin{aligned} \delta^*\phi(x,t) &= \overline{C}(x,t)\epsilon \\ \delta^*C(x,t) &= -\left(\dot\phi(x,t) - \lambda(x,t)\right)\epsilon \\ \delta^*\overline{C}(x,t) &= 0 \\ \delta^*\lambda(x,t) &= \dot{\overline{C}}(x,t)\epsilon \end{aligned} \tag{2.7}$$

where ϵ is a Grassman variable. This symmetry guarantees that the correct equilibrium limit for the correlation functions (2.3)[14,15] is attained.

Introducing a notation in terms of superfields Φ

$$\Phi(z) = \phi(x,t) + \bar\theta\, C(x,t) + \overline{C}(x,t)\,\theta + \lambda(x,t)\,\bar\theta\theta \tag{2.8}$$

where $z \equiv (x,t,\bar\theta,\theta)$ and $(t,\bar\theta,\theta)$ is a superspace with anticommuting coordinates θ and $\bar\theta$ which satisfy

$$\int d\theta\, d\bar\theta\, 1 = \int d\theta\, d\bar\theta\,\theta = \int d\theta\, d\bar\theta\,\bar\theta = 0 \text{ and } \int d\theta\, d\bar\theta\,\bar\theta\theta = 1 \tag{2.9}$$

then the BRS symmetry can be viewed as a translation $\bar\theta \to \bar\theta + \bar\epsilon$ in the superspace. So (2.6) is obtained from

$$\delta\Phi = \Phi(x,t,\bar\theta+\bar\epsilon,\theta) - \Phi(x,t,\bar\theta,\theta) = \bar\epsilon\frac{\partial}{\partial\bar\theta}\Phi \tag{2.10}$$

whereas the anti-BRS symmetry is interpreted as a translation $\theta \to \theta + \epsilon$ and $t \to t - \bar\theta\epsilon$, in a way that (2.7) results from

$$\delta^*\Phi = \Phi(x,t-\bar\theta\epsilon,\bar\theta,\theta+\epsilon) - \Phi(x,t,\bar\theta,\theta) = \epsilon\left(\frac{\partial}{\partial\theta} - \bar\theta\frac{\partial}{\partial t}\right)\Phi \tag{2.11}$$

With the aid of (2.8) the action (2.5) can be written in a supersymmetric form[12,13,16] similarly to the stationary stochastic process, as realized by Parisi and Sourlas[10]. In this situation the functional generator (2.4) assumes the form

$$Z_{SS}[\mathcal{J}] = \int \mathcal{D}\Phi \; e^{-S_{SS}[\Phi] - \int dz\, \mathcal{J}\Phi} \tag{2.12}$$

where

$$S_{SS}[\Phi] = \int dz \, \left(-\Phi\, \overline{D} D\, \Phi + \mathcal{L}[\Phi]\right) \tag{2.13}$$

and

$$\mathcal{J}(z) = K(x,t) + \overline{\vartheta}(x,t)\,\theta + \overline{\theta}\,\vartheta(x,t) + J(x,t)\,\overline{\theta}\theta$$

with K, $\overline{\vartheta}$, ϑ and J as sources to the fields λ, C, $\overline{C}$ and ϕ respectively.

D and $\overline{D}$ are covariant derivatives

$$D \equiv \frac{\partial}{\partial \overline{\theta}} - \theta\, \frac{\partial}{\partial t} \tag{2.14a}$$

$$\overline{D} \equiv \frac{\partial}{\partial \theta} \tag{2.14b}$$

which obey

$$D^2 = \overline{D}^2 = 0 \qquad \text{and} \qquad \{D, \overline{D}\} = -\frac{\partial}{\partial t} \tag{2.15}$$

and $\mathcal{L}[\Phi]$ is a Lagrangian density of the theory with ϕ replaced by the superfield Φ.

Observing that

$$D\overline{D} = \frac{\partial^2}{\partial \overline{\theta} \partial \theta} - \theta\, \frac{\partial^2}{\partial t \partial \theta} \tag{2.16}$$

the integrations on the Grassmann variables θ and $\overline{\theta}$ can be performed through

$$\int d\theta\, d\overline{\theta}\, F(\Phi) = -D\overline{D}\, F\Big|_{\theta=\overline{\theta}=0} \tag{2.17}$$

Also it is easy to demonstrate the following useful relations

$$\begin{aligned} \Phi\Big|_{\theta=\overline{\theta}=0} &= \phi(x,t) \\ D\,\Phi\Big|_{\theta=\overline{\theta}=0} &= C(x,t) \\ -\overline{D}\,\Phi\Big|_{\theta=\overline{\theta}=0} &= \overline{C}(x,t) \\ -D\overline{D}\,\Phi\Big|_{\theta=\overline{\theta}=0} &= \lambda(x,t) \\ \overline{D}D\,\Phi\Big|_{\theta=\overline{\theta}=0} &= \lambda(x,t) - \dot{\phi}(x,t) \end{aligned} \tag{2.18}$$

With the aid of (2.17) and (2.18) is straightforward to get (2.5) from (2.13).

The calculation of correlation functions in (2.3) by (2.13) allows us to interpret the SQM as a superfield theory. The usefulness of (2.13) for the SQM provides a wide range of advantages: Perturbatively it is more economical since it avoids the great number of graphics characteristic of the Langevin approach and of the functional integral given in terms of components fields as in (2.5). The explicit supersymmetry (via the anti-BRS symmetry (2.7)) guarantees the right equilibrium limit to the correlation functions via the dimensional reduction mechanism of Parisi and Sourlas[10,14] adapted to the stochastic case[15]. In the $1/N$ implementation for the NLSM, for instance, this observation implies that the constraint must be imposed supersymmetrically in (2.13). In other words the supersymmetry works as a guide offering a systematic procedure to implement the $1/N$ expansion resembling the techniques common to the equilibrium. Finally, and most important, the renormalization of the theory is more straightforward when we use the supersymmetric formalism. In Ref. 3, for scalar theories, Zinn-Justin used explicitly the BRS symmetry to demonstrate the stability of the action (2.5) by the renormalization. Perturbatively, however, a series of cancellations between the bosonic and the ghost sectors of (2.5) led to the vanishing of counterterms which apparently would destroy the renormalizability of the theory. In a supersymmetric perturbative framework (see Ref. 9) these cancellations are automatic.

3. $1/N$ Expansion of the non Linear Sigma Model

The Lagrangian density for the $O(N)$ NLSM is given by

$$\mathcal{L} = \frac{1}{2}\,(\partial_\mu \phi_a)^2 \tag{3.1}$$

with the constraint

$$\phi_a^2 = \frac{N}{2f} \tag{3.2}$$

The sum over the flavor index a runs from 1 to N and we use the summation convention for repeated indices.

The implementation of the $1/N$ for the NLSM in the SQM can be done as in the static situation. The supersymmetric functional generator (2.12) for (3.1) and (3.2) reads

$$Z_{\mathrm{SS}} = \int \mathcal{D}\Phi\, \delta\left(\Phi_a^2 - \frac{N}{2f}\right) \mathrm{e}^{-\int dz \left[\Phi_a \left(-\overline{D}D - \frac{1}{2}\Box + \frac{1}{2}m^2\right)\Phi_a + \mathcal{J}_a \Phi_a\right]} \tag{3.3}$$

where

$$\mathcal{J}_a = K_a(x,t) + \overline{\vartheta}_a(x,t)\,\theta + \overline{\theta}\,\vartheta_a(x,t) + J_a(x,t)\,\overline{\theta}\theta \tag{3.4}$$

In (3.3) we have used (2.13), replacing ϕ_a by the superfield Φ_a with components

$$\Phi_a = \phi_a + \overline{\theta}\, C_a + \overline{C}_a\, \theta + \lambda_a\, \overline{\theta}\theta \tag{3.5}$$

in (3.1) and (3.2). A mass term for the Φ_a field has been added. At this point the supersymmetric formulation of SQM already exhibits its advantages in the implementation of the $1/N$ expansion. The constraint was taking into account easily and respecting the supersymmetry such that the correct equilibrium limit is reached. Also the $O(N)$ symmetry is manifest in (3.3).

Now we write the delta function constraint as

$$\delta\left(\Phi_a^2 - \frac{N}{2f}\right) = \int \mathcal{D}\Xi\; \mathrm{e}^{\int dz\, \frac{1}{\sqrt{N}} \Xi \left(\Phi_a^2 - \frac{N}{2f}\right)} \tag{3.6}$$

where Ξ is an auxiliary superfield with components

$$\Xi = \sigma + \overline{\theta}\, c + \overline{c}\, \theta + \alpha\, \overline{\theta}\theta \tag{3.7}$$

Substituting (3.6) in (3.3) we get

$$Z_{\mathrm{SS}} = \int \mathcal{D}\Phi\, \mathcal{D}\Xi\; \mathrm{e}^{-\int dz \left[\frac{1}{2}\Phi_a \left(-2\overline{D}D - \Box + m^2\right)\Phi_a - \frac{1}{\sqrt{N}}\Xi\Phi_a^2 + \frac{\sqrt{N}}{2f}\Xi + \mathcal{J}_a\Phi_a\right]} \tag{3.8}$$

The Feynman rules for the $1/N$ expansion of this Lagrangian are easily derived by the same standard procedures of the static case[17]. The superfield propagator for Φ is given by inverting the quadratic form in (3.8). For periodic boundary conditions it must satisfy

$$\left([D,\overline{D}] - \Box + m^2\right) \Delta_{\Phi_a\Phi_b}(z - z') = \delta_{ab}\,\delta(z - z') \tag{3.9}$$

From

$$[D,\overline{D}]^2 = \frac{\partial^2}{\partial t^2} \tag{3.10}$$

we arrive at

$$\Delta_{\Phi_a\Phi_b}(z - z') = \frac{2D\overline{D} + \frac{\partial}{\partial t} - (-\Box + m^2)}{\frac{\partial^2}{\partial t^2} - (-\Box + m^2)^2}\, \delta_{ab}\, \delta(z - z') \tag{3.11}$$

In terms of a Fourier representation it may be written as

$$\Delta_{\Phi_a\Phi_b}(k,\omega,\theta,\overline{\theta}) = \frac{-2D\overline{D} - i\omega + (k^2 + m^2)}{\omega^2 + (k^2 + m^2)^2}\, \delta_{ab}\, \delta(\overline{\theta})\delta(\theta) \tag{3.12}$$

where $D\overline{D} = D\overline{D}(\omega,\theta,\overline{\theta})$.

After the Gaussian integration for Φ in the expression (3.8) and except from constants the effective action reads

$$\mathcal{S}_{\mathrm{eff}} = \frac{N}{2}\mathrm{Str}\ln\left[1 - \frac{2}{\sqrt{N}} \frac{\Xi}{-2\overline{D}D - \Box + m^2}\right] + \frac{\sqrt{N}}{2f}\int dz\, \Xi(z) \tag{3.13}$$

where "Str" is the supertrace[18]. An expansion in powers of $1/\sqrt{N}$ for the effective action

$$\mathcal{S}_{\mathrm{eff}} = \sqrt{N}\mathcal{S}^{(1)} + \mathcal{S}^{(2)} + \mathcal{O}\left(\frac{1}{\sqrt{N}}\right) \tag{3.14}$$

yields

$$\mathcal{S}^{(1)} = -\mathrm{Str}\left(\frac{\Xi}{-2\overline{D}D - \Box + m^2}\right) + \frac{1}{2f}\int dz\, \Xi(z) \tag{3.15}$$

The existence of the saddle-point implies that in the effective action (3.13) no linear term in Ξ should exist, so we require that

$$\begin{aligned} \int d\theta d\overline{\theta}\, \Xi(0,0,\theta,\overline{\theta}) \Bigg\{ \frac{1}{2f} - \int \frac{d^n k\, d\omega}{(2\pi)^{n+1}} \int d\theta' d\overline{\theta}'\, \delta(\overline{\theta} - \overline{\theta}')\delta(\theta - \theta') \\ \times \left[\frac{-2D\overline{D}(\omega,\theta',\overline{\theta}') - i\omega + (k^2 + m^2)}{\omega^2 + (k^2 + m^2)^2}\right] \delta(\overline{\theta} - \overline{\theta}')\delta(\theta - \theta') \Bigg\} = 0 \end{aligned} \tag{3.16}$$

Performing the integrations we finally arrive, for $n = 2$, at

$$\int \frac{d^2 k}{(2\pi)^2}\, \frac{1}{k^2 + m^2} - \frac{1}{2f} = 0 \tag{3.17}$$

This is the usual mass-gap formula and defining a renormalized coupling constant by

$$\frac{1}{f_R(\mu)} = \frac{1}{f} + \frac{1}{2\pi}\ln\frac{\Lambda^2}{\mu^2} \tag{3.18}$$

where Λ is a Pauli-Villars regularization mass and μ the renormalization point, it follows that

$$m^2 = \mu^2 e^{-\frac{2\pi}{f_R}} \tag{3.19}$$

The term $S^{(2)}$ gives us the quadratic part in Ξ

$$S^{(2)} = -\mathrm{Str}\left(\frac{\Xi}{-2\overline{D}D - \Box + m^2}\right)^2 \tag{3.20}$$

which after standard manipulations yields

$$\begin{aligned}
&-\int \frac{d^n k d\omega}{(2\pi)^{n+1}} \int d\theta d\overline{\theta} d\theta' d\overline{\theta}'\, \Xi(-k,-\omega,\theta,\overline{\theta})\,\Xi(k,\omega,\theta',\overline{\theta}') \\
&\times \int \frac{d^n k' d\omega'}{(2\pi)^{n+1}} \frac{-2D\overline{D}(\omega'+\omega,\theta,\overline{\theta}) - i(\omega'+\omega) + [(k'+k)^2+m^2]}{(\omega'+\omega)^2 + [(k'+k)^2+m^2]^2}\,\delta(\overline{\theta}-\overline{\theta}')\delta(\theta-\theta') \\
&\times \frac{-2D\overline{D}(\omega',\theta',\overline{\theta}') - i\omega' + (k'^2+m^2)}{\omega'^2 + (k'^2+m^2)^2}\,\delta(\overline{\theta}'-\overline{\theta})\delta(\theta'-\theta)
\end{aligned} \tag{3.21}$$

Applying the techniques described in Ref. 18 we integrate $D\overline{D}(\omega'+\omega,\theta,\overline{\theta})$ by parts to perform the θ', $\overline{\theta}'$ integrations. We can write, therefore

$$-\frac{1}{2}\int \frac{d^n k d\omega}{(2\pi)^{n+1}} \int d\theta d\overline{\theta}\, \Xi(-k,-\omega,\theta,\overline{\theta})\, \{4A(k,\omega)[D,\overline{D}] + 8B(k,\omega)\}\, \Xi(k,\omega,\theta,\overline{\theta}) \tag{3.22}$$

where

$$A(k,\omega) = \int \frac{d^n k' d\omega'}{(2\pi)^{n+1}} \frac{1}{\omega'^2 + (k'^2+m^2)^2} \frac{1}{(\omega'+\omega)^2 + [(k'+k)^2+m^2]^2} \tag{3.23a}$$

$$B(k,\omega) = \int \frac{d^n k' d\omega'}{(2\pi)^{n+1}} \frac{-i\omega' + (k'^2+m^2)}{\omega'^2 + (k'^2+m^2)^2} \frac{1}{(\omega'+\omega)^2 + [(k'+k)^2+m^2]^2} \tag{3.23b}$$

These integrals can be explicitly calculated for $n = 2$, as we already have shown in I, and they yield

$$\begin{aligned}
A(k,\omega) = \frac{1}{8\pi}\frac{i}{\omega}\Bigg\{ &\frac{1}{\sqrt{R^2+4k^2m^2}} \ln\left[\frac{R-k^2-\sqrt{R^2+4k^2m^2}}{R-k^2+\sqrt{R^2+4k^2m^2}}\, \frac{R+\sqrt{R^2+4k^2m^2}}{R-\sqrt{R^2+4k^2m^2}}\right] \\
&-(R\to -S)\Bigg\}
\end{aligned} \tag{3.24a}$$

$$B(k,\omega) = \frac{1}{8\pi}\frac{1}{\sqrt{S^2+4k^2m^2}} \ln\left[\frac{S+k^2+\sqrt{S^2+4k^2m^2}}{S+k^2-\sqrt{S^2+4k^2m^2}}\, \frac{S-\sqrt{S^2+4k^2m^2}}{S+\sqrt{S^2+4k^2m^2}}\right] \tag{3.24b}$$

where

$$R \equiv k^2 + i\omega$$

$$S \equiv -k^2 + i\omega$$

Finally from (3.22) the propagator $\Delta_{\Xi\Xi}$ can be calculated, hence

$$\begin{aligned}
\Delta_{\Xi\Xi}(k,\omega,\theta,\overline{\theta}) &= -\frac{1}{4A(k,\omega)[D,\overline{D}] + 8B(k,\omega)}\,\delta(\overline{\theta})\delta(\theta) \\
&= \frac{1}{4}\frac{A(k,\omega)(D\overline{D}+i\omega) - 2B(k,\omega)}{\omega^2A^2(k,\omega) + 4B^2(k,\omega)}\,\delta(\overline{\theta})\delta(\theta)
\end{aligned} \tag{3.25}$$

The stochastic super-Feynman rules for the $1/N$ expansion of the NLSM are summarized in Fig. 1. The graphs of Fig. 2 are illegal as subgraphs since they have already been taken into account in the gap equation (3.17) and in the effective propagator (3.25).

Figure 1

Figure 2

At this point we demonstrate that (3.25) allows us to recover the propagator for the σ field in the equilibrium . From (3.7) we have

$$\Delta_{\sigma\sigma}(k,\omega) = \Delta_{\Xi\Xi}(k,\omega,\theta-\theta',\overline{\theta}-\overline{\theta}')\Big|_{\substack{\theta=\overline{\theta}=0\\ \theta'=\overline{\theta}'=0}} \tag{3.26}$$

and the static limit can be reached through an integration in ω

$$\Delta_{\sigma\sigma}(k) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\,\Delta_{\sigma\sigma}(k,\omega) \tag{3.27}$$

Instead of perform the integration directly we can use the fluctuation dissipation theorem (see the Appendix for a derivation) in its form given by (A.8). In our case the response function is

$$\Delta_{\sigma\sigma}(k,\omega) = -D\overline{D}\Delta_{\Xi\Xi}(k,\omega,\theta-\theta',\overline{\theta}-\overline{\theta}')\Big|_{\substack{\theta=\overline{\theta}=0\\ \theta'=\overline{\theta}'=0}} \tag{3.28}$$

and (A.8) yields

$$\Delta_{\sigma\sigma}(k) = \Delta_{\sigma\sigma}(k,\omega=0) \tag{3.29}$$

and finally

$$\Delta_{\sigma\sigma}(k) = -\frac{1}{8B(k,\omega=0)} \tag{3.30}$$

For $n=2$ we substitute the result (3.24b) and

$$\Delta_{\sigma\sigma}(k) = -\frac{1}{\frac{1}{\pi}\frac{1}{\sqrt{k^2(k^2+4m^2)}}\ln\left[\frac{\sqrt{k^2+4m^2}+\sqrt{k^2}}{\sqrt{k^2+4m^2}-\sqrt{k^2}}\right]} \tag{3.31}$$

which is the usual result of the equilibrium[17].

4. Divergencies in Super-Feynman Graphs and Renormalizability

The super-Feynman rules given in Fig. 1 allows us to calculate the degree of divergence of any super-Feynman graph. The analytical form of these super-Feynman diagrams is given by

$$\int\prod_{i=1}^{L}\frac{d^nk_id\omega_i}{(2\pi)^{n+1}}\int\prod_{j=1}^{V}d\theta_jd\overline{\theta}_j\prod_{\ell=1}^{N_{I\Phi}}\Delta_{\Phi\Phi}(K_\ell,\Omega_\ell,\theta_{\ell_1}-\theta_{\ell_2},\overline{\theta}_{\ell_1}-\overline{\theta}_{\ell_2})$$
$$\times\prod_{\ell'=1}^{N_{I\Xi}}\Delta_{\Xi\Xi}(K_{\ell'},\Omega_{\ell'},\theta_{\ell'_1}-\theta_{\ell'_2},\overline{\theta}_{\ell'_1}-\overline{\theta}_{\ell'_2}) \tag{4.1}$$

where L is the number of loops, V is the number of vertices, $N_{I\Phi}$ is the number of internal Φ lines representing the superpropagator $\Delta_{\Phi\Phi}$ and $N_{I\Xi}$ is the number of internal Ξ lines representing the superpropagator $\Delta_{\Xi\Xi}$. In the $\ell(\ell')$th $\Phi(\Xi)$ line, joining the vertices with $\theta_{\ell_1},\overline{\theta}_{\ell_1}(\theta_{\ell'_1},\overline{\theta}_{\ell'_1})$ and $\theta_{\ell_2},\overline{\theta}_{\ell_2}(\theta_{\ell'_2},\overline{\theta}_{\ell'_2})$ flows momentum $K_\ell,\Omega_\ell(K_{\ell'},\Omega_{\ell'})$, which is a linear

combination of external and loop momentum k_i,ω_i. Replacing k_i by λk_i in (4.1) and considering $\lambda\to\infty$, we get for the superficial degree of divergence d, of the super-Feynman stochastic diagrams, the expression

$$d = (n+2)L + 2(V-1) - 4N_{I\Phi} - (n-2)N_{I\Xi}$$

where we have made use of

$$\Delta_{\Phi\Phi}(\lambda k,\lambda^2\omega,\lambda^{-1}\theta,\lambda^{-1}\overline{\theta}) = \lambda^{-4}\Delta_{\Phi\Phi}(k,\omega,\theta,\overline{\theta}) \tag{4.2a}$$

$$\Delta_{\Xi\Xi}(\lambda k,\lambda^2\omega,\lambda^{-1}\theta,\lambda^{-1}\overline{\theta}) = \lambda^{-(n-2)}\Delta_{\Xi\Xi}(k,\omega,\theta,\overline{\theta}) \tag{4.2b}$$

and that the maximum number of independent differences of θ's and differences of $\overline{\theta}$'s one can construct is $V-1$. Since we have the topological relations $L = N_I - V + 1$, $V = N_{E\Xi} + 2N_{I\Xi}$ and $2V = N_{E\Phi} + 2N_{I\Phi}$ where N_I is the total number of internal lines, $N_{E\Phi}$ is the number of external Φ lines and $N_{E\Xi}$ the number of external Ξ lines, it follows that

$$\delta = n - \frac{n-2}{2}N_{E\Phi} - 2N_{E\Xi} \tag{4.3}$$

Note that this degree of divergence is equal to the static case if we replace $N_{E\Phi}$ and $N_{E\Xi}$ by $N_{E\phi}$ and $N_{E\sigma}$ respectively. For $n=2$ we see that δ does not depend on the number of external Φ-lines and the theory is apparently nonrenormalizable since an infinity number of counterterms are necessary to absorb these divergencies. In I it was proved that the NLSM is renormalizable in $2\le n<4$ dimensions in the component formulation, given by (2.5), of the SQM. The key elements of that proof were a BRS identity plus graphical Ward identities which kept the form of the action invariant by the renormalization process. However, in the present superfield formulation we have a direct demonstration due to the graphical identity of Fig. 3 easily obtained by our super-Feynman rules. This is the stochastic generalization of the Aref'eva identity[4].

Figure 3

From this identity the non-Lagrangian counterterms will compensate each other. Thus, the sum of the divergent parts of diagrams with the same order of $1/N$ as, for example, those in Fig. 4 corresponding to a Φ^4 counterterm, is equal to zero. The graph in Fig. 4b has a subgraph with the same divergence as the graph in Fig. 4a. Contracting this subgraph to a point we get the cancellation of these divergencies thanks to the identity of Fig. 3. It is easy to convince oneself that this mechanism gets ride of all non-Lagrangian counterterms.

Figure 4 Figure 5

In this way we just have to analyse the divergence of one-particle irreducible graphs with $N_{E\Phi} \leq 2$ depicted in Fig. 5. The diagram in Fig. 5a is quadratically divergent while the ones in Fig. 5b and 5c are logarithmic divergent. Therefore, the theory can be made ultraviolet finite, after an appropriate regularization, by a renormalized action obtained from (3.8) through the following redefinitions

$$
\begin{aligned}
\Phi(w,\omega,\theta,\bar\theta) &= Z_{\Phi}^{1/2}\Phi_R(w,\omega,\theta,\bar\theta)\\
\Xi(w,\omega,\theta,\bar\theta) &= Z_{\Xi}^{1/2}\Xi_R(w,\omega,\theta,\bar\theta)\\
m^2 &= m_R^2 + \frac{\delta m^2}{Z_{\Phi}}\\
f &= Z_f f_R\\
t &= Z_t t_R\\
\theta &= Z_t^{1/2}\theta_R\\
\bar\theta &= Z_t^{1/2}\bar\theta_R
\end{aligned}
\tag{4.4}
$$

where Φ, Ξ, m, f, t, θ and $\bar\theta$ are the bare parameters of the theory. Obviously from (2.16) it follows that $\overline{D}D = Z_t^{-1}\overline{D}_R D_R$. The renormalization constants Z_{Φ},Z_{Ξ}, δm^2, Z_f and Z_t can be calculated in perturbation theory using, for instance, a minimal subtraction scheme.

An explicit calculation of the renormalization constants should yield the static ones except Z_t which is related with the dynamics of the stochastic process. In fact, due to the analytic form of our propagator (3.25) it is rather involved to obtain the divergent parts of our super-Feynman graphs, for instance, in a dimensional regularization scheme. Progress is being done in this direction and will be reported elsewhere.

5. Conclusion

We have developed, in some detail, the $1/N$ expansion of field theories in the supersymmetric functional approach for the stochastic quantization of Parisi and Wu. We have used as an example the non linear sigma model in two dimensions to implement the superperturbation technique.

We have shown how this procedure resembles the equilibrium one mainly in the aspects related to its renormalization. In this case the renormalization is set up more easily than in the component formulation given in I. The systematic developed here for the $1/N$ expansion can be extended without further difficulties for the $\lambda\phi^4$, Gross–Neveu models and CP^N models.

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Appendix

Fluctuation Dissipation Theorem[13,19]

The Ward–Takahashi identities associated with the symmetries (2.10) and (2.11) can be easily derived . Since $\mathcal{S}_{SS}$ is invariant under $\Phi \to \Phi + (\delta + \delta^*)\Phi$ and that $Z_{SS}[\mathcal{J}]$ does not change under the changes of variables the following identity can be derived

$$\delta Z_{SS}[\mathcal{J}] = \left(\int dz\, \mathcal{J}(z)\mathcal{O}(z)\frac{\delta}{\delta\mathcal{J}(z)}\right) Z_{SS}[\mathcal{J}] = 0 \tag{A.1}$$

where $\mathcal{O} \equiv \delta + \delta^*$. This Ward-Takahashi identity allows us to deduce relations among correlations functions. From $\frac{\delta^2}{\delta\mathcal{J}(z)\delta\mathcal{J}(z')}\delta Z_{SS}[\mathcal{J}]\Big|_{\mathcal{J}=0} = 0$ we get

$$\left\langle \Phi(z)\left[\mathcal{O}(z')\Phi(z')\right]\right\rangle = -\left\langle \left[\mathcal{O}(z)\Phi(z)\right]\Phi(z')\right\rangle \tag{A.2}$$

Using (2.10), (2.11) and (2.14) we write

$$\mathcal{O} = \bar\epsilon(D + \theta\partial_t) + \epsilon\left(\overline{D} - \bar\theta\partial_t\right) \tag{A.3}$$

and since ϵ and $\bar\epsilon$ are independent and arbitrary we see from $(A.2)$ that

$$\left\langle \Phi(z)\left(D' + \theta'\partial_t'\right)\Phi(z')\right\rangle + \left\langle (D + \theta\partial_t)\,\Phi(z)\Phi(z')\right\rangle = 0 \tag{A.4a}$$

$$\left\langle \Phi(z)\left(\overline{D}' - \bar\theta'\partial_t'\right)\Phi(z')\right\rangle + \left\langle (\overline{D} + \theta\partial_t)\,\Phi(z)\Phi(z')\right\rangle = 0 \tag{A.4b}$$

Applying $\overline{D}'$ and D on $(A.4a)$ and $(A.4b)$ respectively, subtracting one from another, and setting $\theta = \bar\theta = \theta' = \bar\theta' = 0$ we have after employing (2.18)

$$\partial_t\left\langle \phi(x,t)\phi(x',t')\right\rangle = \left\langle \lambda(x,t)\phi(x',t')\right\rangle - \left\langle \phi(x,t)\lambda(x',t')\right\rangle \tag{A.5}$$

This is the fluctuation dissipation theorem which relates the correlation function $\langle\phi(x,t)\phi(x',t')\rangle$ to the response function $\langle\lambda(x,t)\phi(x',t')\rangle$. Through a Fourier transform we can rewrite the above equation as

$$\Delta_{\phi\phi}(k,\omega) = \frac{2}{\omega}\,\mathrm{Im}\,\Delta_{\lambda\phi}(k,\omega) \tag{A.6}$$

where we have taken into account $\Delta^*_{\lambda\phi}(k,\omega) = \Delta_{\lambda\phi}(k,-\omega)$. Now from the causal nature of the response function

$$\Delta_{\lambda\phi}(k,\omega) = \int\frac{d\omega'}{\pi}\,\frac{\mathrm{Im}\,\Delta_{\lambda\phi}(k,\omega')}{\omega' - \omega} \tag{A.7}$$

Eq. $(A.6)$ implies

$$\Delta_{\lambda\phi}(k,\omega = 0) = \int\frac{d\omega'}{2\pi}\,\Delta_{\phi\phi}(k,\omega') \tag{A.8}$$

which simply says that the static correlation function can be calculated from the response function at $\omega = 0$. As a simple application of the fluctuation dissipation theorem consider a free scalar theory given by $\frac{\delta S}{\delta\phi} = (-\Box + m^2)\phi$ in (2.5), then

$$\Delta_{\phi\phi}(k,\omega) = \frac{2}{\omega^2 + (k^2 + m^2)^2} \tag{A.9a}$$

$$\Delta_{\lambda\phi}(k,\omega) = \frac{1}{-i\omega + (k^2 + m^2)} \tag{A.9b}$$

where it is worthwhile to observe that $\Delta_{\lambda\phi}$ is simply the Fourier transformed Green function of the Langevin equation (2.1). The equilibrium result $\frac{1}{k^2+m^2}$ follows straightforwardly from $\Delta_{\lambda\phi}(k,\omega = 0)$.

References

1. J. C. Brunelli and M. Gomes, *Phys. Rev.* **D46**, 2617 (1992).
2. S. Coleman, *1/N*, Pointlike Structures Inside and Outside Hadrons (Ed. Zichichi), Plenum, New York (1981).
3. J. Zinn-Justin, *Nucl. Phys.* **B275[FS17]**, 135 (1986).
4. I. Ya. Aref'eva, *Theor. Math. Phys.* **36**, 573 (1979).
5. G. Parisi and Y. Wu, *Sci. Sinica* **24**, 483 (1981).
6. P. H. Damgaard and H. Hüffel, *Phys. Rep.* **152**, 227 (1987).
7. C. M. Bender, F. Cooper and B. Freedman, *Nucl. Phys.* **B219**, 61 (1983); Y. Nakano, *Prog. Theor. Phys.* **69**, 361 (1983); E. Gozzi, *Phys. Rev.* **D28**, 1922 (1983).
8. C. Becchi, A. Rouet and R. Stora, *Phys. Lett.* **52B**, 344 (1974); C. Becchi, A. Rouet and R. Stora, *Ann. Phys. (N.Y.)* **98**, 287 (1976).
9. E. Sh. Egoryan and R. P. Manvelyan, *Theor. Math. Phys.* **68**, 762 (1986); S. Marculescu, K. Okano and L. Schülke, *Nucl. Phys.* **B349**, 463 (1991).
10. G. Parisi and N. Sourlas, *Phys. Rev. Lett.* **43**, 744 (1979); *Nucl. Phys.* **B206**, 321 (1982).
11. H. Nakazato, M. Namiki, I. Ohba and K. Okano, *Prog. Theor. Phys.* **70**, 298 (1983).
12. E. Egoryan and S. Kalitzin, *Phys. Lett.* **129B**, 320 (1983).
13. M. V. Felgel'man and A. M. Tsvelik, *Sov. Phys. JETP* **56**, 823 (1982).
14. J. L. Cardy, *Phys. Lett.* **125B**, 470 (1983); A. Klein and J. F. Perez, *Phys. Lett.* **125B**, 473 (1983).
15. R. Kirschner, *Phys. Lett.* **139B**, 180 (1984); E. Gozzi, *Phys. Lett.* **B143**, 183 (1984).
16. S. Chaturvedi, A. K. Kapoor and V. Srinavasan, *J. Phys.* **A17**, 2037 (1984).
17. C. I. Rim and W. I. Weisberger, *Phys. Rev.* **D30**, 1763 (1984) and references therein.
18. S. J. Gates, M. T. Grisaru, M. Rocěc and W. Siegel, Superspace (Benjamin–Cummings, New York, 1983); P. West, Introduction to Supersymmetry and Supergravity (World Scientific, Singapore, 1986).
19. S. Chaturvedi, A. K. Kapoor and V. Srinavasan, *Z. Phys.* **C57**, 249 (1984); E. Gozzi, *Phys. Rev.* **D30**, 1218 (1984).

Figure Captions

Figure 1 – Super-Feynman rules for the $1/N$ expansion.

Figure 2 – Illegal supersubgraphs.

Figure 3 – Aref'eva mechanism of cancellation for insertions of Φ fields.

Figure 4 – Diagrams with compensating divergent parts.

Figure 5 – Superficially divergent diagrams.

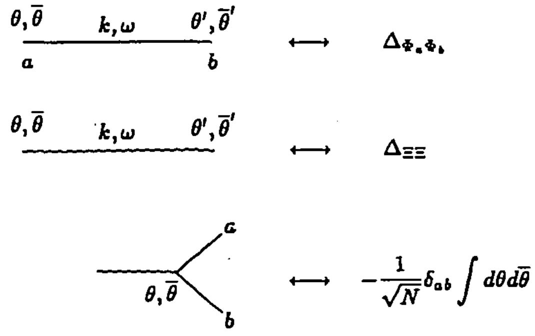

Figure 1

Figure 2

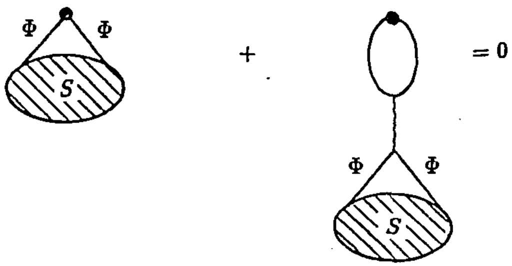

Figure 3

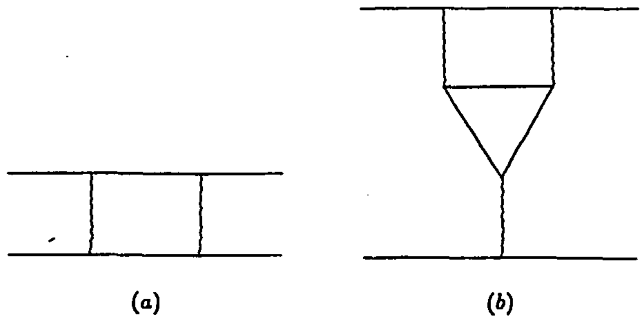

Figure 4

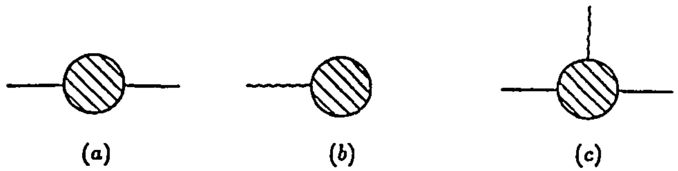

Figure 5